Stability Analysis and Robust Composite Controller Synthesis for Flexible Joint Robots

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Abstract

In this paper the control of flexible joint manipulators is studied in detail. The model of N–axis flexible joint manipulators are derived and reformulated in the form of singular perturbations, and integral manifold is used to separate fast dynamics from slow dynamics. A composite control algorithm is proposed for the flexible joint robots, which consists of two main parts. Fast control, u_f, which guarantees that the fast dynamics remains asymptotically stable, and the corresponding integral manifold remains invariant. Slow control ,u_s, itself consists of a robust PID designed based on the rigid model, and a corrective term designed based on the reduced flexible model. The stability of the fast dynamics, robust stability of the PID scheme are analyzed separately, and finally, the closed–loop system is proved to be UUB stable, by Lyapunov stability analysis. Finally, the effectiveness of the proposed control law is verified through simulations. The simulation results of single and two–link flexible joint manipulators are compared to that given in the literature. It is shown that the proposed control law ensure the robust stability and performance, despite the modeling uncertainties.

Keywords

Flexible joint robots, harmonic drive, singular perturbation, integral manifold, reduced flexible model, robust composite control, robust PID, UUB Stability, Lyapunov analysis, simulation verification.

I. Introduction

Multiple–axis robot manipulators are widely used in industrial and space applications. The success in reaching high accuracy in these robots is due to their rigidity, which make them highly controllable. After the inception of harmonic drive in 1955, and its wide acceptance, and use in the design of many electrically–driven robots, the rigidity of the robot manipulators are affected greatly. In early eighties researchers showed that the use of control algorithms developed based on rigid robot dynamics on real non–rigid robots is very limited and may even cause instability [18]. The singular perturbation theory is used as the basis to model the dynamics of the flexible joint robots, in which by use of two–time scale behavior, these systems are divided into fast and slow subsystems[9], [10]. As shown in [2] for a three–axis flexible robot the system is not feedback linearizable, and the use of methods such as computed–torque methods for flexible manipulators is not directly implementable. By neglecting the effects of link motion on the kinetic energy of the rotor, Spong has derived a mathematical model for such systems, in which the system is feedback linearizable [16]. However, to linearize the system acceleration and jerk feedback is required whose measurement are costly. To avoid the need of acceleration and jerk in this method the idea of integral manifold is employed. In this method instead of using the zero order approximation of the model extracted from the singular perturbation theory, higher order models can be used, and hence, a series of corrective terms is added to the control algorithm [6], [18]. In adaptive methods many algorithms are developed for FJR's, in most of which a term due to the fast subsystem is added to the adaptive algorithm based on rigid models [5], [6]. In robust methods by considering model uncertainties the stability of the fast subsystem is first analyzed and by the use of robust control synthesis, a robust controller is designed for the slow subsystem [1], [3].

As it is shown, most of the research on FJR's are concentrated on nonlinear control schemes. In this paper we propose a new method based on the simple form of PID, and analyze the robust stability of the uncertain closed–loop system in the presence of structured and unstructured uncertainties. In this analysis we borrow the idea of the singular perturbation model of the flexible joint robot, but in presence of the modeling uncertainties, and divide the system into slow and fast subsystems. Then we introduce an integral manifold plus a composite control law in order to restrain the integral manifold invariant and to satisfy asymptotic stability requirement. The control effort consists of three elements, the first element is designed for the fast subsystem, the second term is a robust PID control designed for the rigid subsystem and the third term is a corrective term designed based on the first order approximation of the reduced flexible system. Based on the Lyapunov stability theory the complete closed–loop system is proven to be UUB stable. In order to verify the effectiveness of the proposed control law design, and compare its results to that presented in the literature, simulation of single and two link flexible joint manipulators are examined. It is shown in this study that the proposed control law ensure the robust stability and performance, despite the modeling uncertainties.

II. Flexible Joint Robot Modeling

Spong [16], has derived a nonlinear dynamical model for FJR using singular perturbation, in which the slow states are the position and velocities of the joints and the fast states are the forces and their derivatives. In order to model an N–axis robot manipulator with n revolute joints assume that: $\hat{q}_i : i = 1, 2, ..., n$ denote the position of i'th link and $\hat{q}_i : i = n+1, n+2, ..., 2n$ denote the position of the i'th actuator scaled by the actuator gear ratio. If the joint is rigid $\hat{q}_i = \hat{q}_{n+i} \forall i$. For flexible joint, if the flexibility is modeled with a linear torsional spring with constant k_i, the elastic force z_i is derived from:

$$z_i = k_i(\hat{q}_i - \hat{q}_{n+i}) \tag{1}$$

The spring constants k_i's are relatively large and rigidity is modeled by the limit $k_i \to \infty$. Let u_i denotes the generalized force applied by the i'th actuator and use the notation:

$$q = (\hat{q}_1, ..., \hat{q}_n, \hat{q}_{n+1}, ..., \hat{q}_{2n})^T = (q_1^T | q_2^T)^T \tag{2}$$

The equation of motion of the system can be written in the following form using Euler–Lagrange

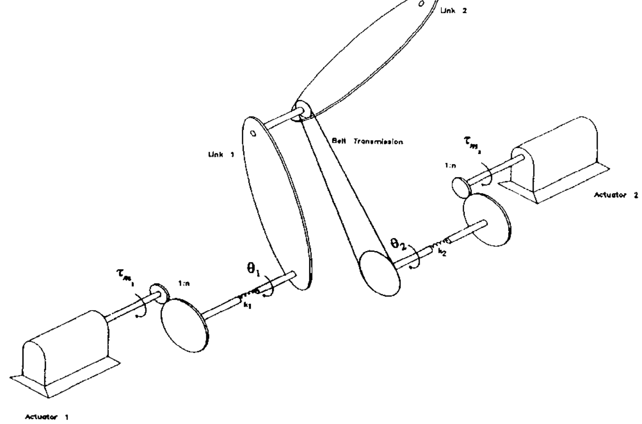

Fig. 1. Two-link Flexible Joint Manipulator.

formulation.

$$\begin{cases} M(q_1)\ddot{q}_1 + N(q_1, \dot{q}_1) \ = \ K(q_2 - q_1) \\ J\ddot{q}_2 \ = \ K(q_1 - q_2) - D\dot{q}_2 + T_F \ + \ u \end{cases} \tag{3}$$

in which,

$$N(q_1, \dot{q}_1) = V_m(q_1\dot{q}_1)\dot{q}_1 + G(q_1) + F_d\dot{q}_1 + F_s(\dot{q}_1) + T_d \tag{4}$$

and K is the joint stiffness matrix, $M(q_1)$ is the mass matrix, $V_m(q_1\dot{q}_1)$ is the matrix of Coriolis and centrifugal terms, $G(q_1)$ is the vector of gravity terms, F_d is the viscous friction matrix, $F_s(\dot{q}_1)$ is the Coulomb friction vector, T_d is the vector of the joint bounded unmodeled dynamics, J is the actuator moments of inertia matrix, D is the actuator viscous friction matrix, and T_F is the actuator bounded unmodeled dynamics. For all revolute manipulators, it is shown in [4], [12]

$$m_1 I \le M(q_1) \le m_2 I \quad ; \quad ||V_m(q_1 \ \dot{q}_1)|| \le \zeta_c ||\dot{q}_1|| \tag{5}$$

$$||G(q_1)|| \le \zeta_g \quad ; \quad ||F_d\dot{q}_1 + F_s(\dot{q}_1)|| \ = \ \zeta_{f0} + \zeta_{f1}||q_1|| \tag{6}$$

$$j_1 I \le J \le j_2 I \quad ; \quad d_1 I \le D \le d_2 I \tag{7}$$

Moreover, if the perturbations are bounded:

$$||T_d|| \le \zeta_e \quad ; \quad ||T_F|| \le \zeta_{f2} \tag{8}$$

in which $\zeta_{f2}, \zeta_e, d_2, d_1, j_2, j_1, \zeta_{f1}, \zeta_{f0}, \zeta_g, \zeta_c, m_2, m_1$ are positive real constants. If the joints are all rigid:

$$M_t(q)\ddot{q} + N_t(q, q) \ = \ u_0 \tag{9}$$

in which $q = q_1$ and M_t is a positive definite matrix. This model is the model of FJR where $k \to \infty$ verifying that the FJR model is a singularly perturbed model of rigid system. Assume that all spring constants are equal[1] the elastic forces of the springs can be calculated by:

$$z \ = \ k(q_1 - q_2)\,,\ K \ = \ kI \tag{10}$$

in order to use a small quantity for singular perturbation define $\epsilon = \frac{1}{k}$ by which for rigid system ($k \to \infty$) in this form we have $\epsilon \to 0$. Multiplying M^{-1} to the both side of 3 and taking

[1]This assumption does not reduces the generality of the formulation, since by scaling z we reach to the same conclusion.

$z = k(q_1 - q_2)$, $q = q_1$, and using $\dot{q}_2 = \dot{q}_1 - \epsilon\dot{z}$:

$$\begin{cases} \ddot{q} = a_1(q,\dot{q}) + A_1(q)z \\ \epsilon\ddot{z} = a_2(q,\dot{q},\epsilon\dot{z}) + A_2(q)z + B_2u \end{cases} \tag{11}$$

in which,

$$A_1 = -M^{-1}(q) \quad ; \quad a_1 = -M^{-1}(q)N(q,\dot{q}) \tag{12}$$

$$a_2 = -\epsilon J^{-1}D\dot{z} + J^{-1}D\dot{q} - J^{-1}T_F - M^{-1}(q)N(q,\dot{q}) \tag{13}$$

$$A_2 = -(M^{-1}(q) + J^{-1}) \; , \; B_2 = -J^{-1} \tag{14}$$

Equation 11 represents FJR as a nonlinear and coupled system. This representation includes both rigid and flexible subsystems in form of a singular perturbation model.

III. Reduced Flexible Model

The singular perturbation model of the FJR is given in Equation 11, This model represents the flexibility in the joints, however, the reduced order model is the model of rigid system, which can be easily derived from Equation 11 by setting $\epsilon = 0$. With some matrix manipulation it can be shown that:

$$(M + J)\ddot{q} + N - T_F + D\dot{q} = u_0$$

Rewrite this equation in this form:

$$M_t(q)\ddot{q} + N_t(q,\dot{q}) = u_0 \tag{15}$$

in which

$$M_t(q) = M(q) + J \tag{16}$$

$$N_t(q,\dot{q}) = N(q,\dot{q}) - T_F + D\dot{q} =$$
$$V_m(q,\dot{q})\dot{q} + G(q) + (F_d + D)\dot{q} + F_s(\dot{q}) + T_d - T_F \tag{17}$$

This representation introduces a 2n dimension Manifold, M_o, which is called the rigid Manifold. If $\epsilon \neq 0$ the produced manifold M_ϵ, which is a function of ϵ represents the flexible system. To define flexible manifold M_ϵ assume:

$$z = H(q,\dot{q},u,\epsilon) \qquad q\epsilon R^n, u\epsilon R^n, z\epsilon R^n \tag{18}$$

$$\dot{z} = \dot{H}(q,\dot{q},u,\epsilon) \qquad q\epsilon R^n, u\epsilon R^n, z\epsilon R^n \tag{19}$$

M_ϵ is an integral manifold for the flexible system if for each initial condition

$$\begin{cases} z(t) = \Delta \\ \dot{z}(t) = \Delta' \end{cases} \text{and} \begin{cases} q(t) = \zeta \\ \dot{q}(t) = \zeta' \end{cases}$$

in M_ϵ all trajectories of $q(t)$ and $z(t)$ for $t > t_o$ remain in the manifold M_ϵ. In other words $\forall t > t_o$:

$$z(t) = H(q(t),\dot{q}(t),u(t),\epsilon) \tag{20}$$

$$\dot{z}(t) = \dot{H}(q(t),\dot{q}(t),u(t),\epsilon) \tag{21}$$

Equations 20 and 21 are called the manifold conditions. An integral manifold for FJR exists if $A_2 = -(M^{-1} + J^{-1})$ is nonsingular $\forall q \in R^n$ [10]. This is always true since the mass matrices M, and J are positive definite. If the manifold condition are not satisfied at initial time t_o, but the fast dynamics are asymptotically stable, the initial transient will die down shortly, and the manifold condition will be satisfied after a short transient.

In order to derive the reduced flexible model, the flexible manifold is used in the formulation. Assume that the function H is several time differentiable with respect to its arguments. Hence, by differentiating Equations 20 and 21 and substitution in Equation 11:

$$\epsilon \ddot{H}(q, \dot{q}, u, \epsilon) = a_2(q, \dot{q}, \epsilon \dot{H}(q, \dot{q}, u, \epsilon)) + A_2(q)H(q, \dot{q}, u, \epsilon) + B_2 u \tag{22}$$

in which,

$$\dot{H} = (\frac{\partial H}{\partial q} + \frac{\partial H}{\partial u}\frac{\partial u}{\partial q})\dot{q} + \frac{\partial H}{\partial \dot{q}}(a_1 + A_1 H) + \frac{\partial H}{\partial u}\frac{\partial u}{\partial t} \tag{23}$$

Now, the reduced flexible model can be derived by replacing $z, \dot{z}$ with $H, \dot{H}$ in Equation 11.

$$\ddot{q} = a_1(q, \dot{q}) + A_1(q)H(q, \dot{q}, u, \epsilon) \tag{24}$$

The order of this equation is equal to the rigid system, however, this model includes the effects of flexibility in form of an invariant integral manifold embedded in itself. Hence, this reduced order model is not an approximation of the FJR model, but it represents its projection on the integral manifold.

IV. Composite Control

In order that the reduced flexible model hold for the system, it is essential that the M_ϵ be an invariant manifold, or the fast dynamics be asymptotically stable. This can be satisfied using a composite control scheme [9]. In this framework the control effort u consists of two main parts, u_s the control effort for slow subsystem, and u_f the control effort for fast subsystem, as:

$$u = u_s(q, \dot{q}, \epsilon) + u_f(\eta, \dot{\eta}) \tag{25}$$

in which $u_f(\eta, \dot{\eta})$ is designed such that the fast dynamics becomes asymptotically stable. η denotes the deviations of fast state variables from the integral manifold.

$$\eta = z - H(q, \dot{q}, u_s, \epsilon) \tag{26}$$

$$\dot{\eta} = \dot{z} - \dot{H}(q, \dot{q}, u_s, \epsilon) \tag{27}$$

The slow component of the control effort, $u_s(q, \dot{q}, \epsilon)$, is also designed based on the reduced flexible model. In this section we describe the design technique for u_f and u_s in the next subsections, respectively.

A. Fast Subsystem Dynamics and Control

Recall Equation 26

$$\epsilon\ddot{\eta} = \epsilon\ddot{z} - \epsilon\ddot{H} =$$
$$a_2(q, \dot{q}, \epsilon\dot{z}) + A_2(q)z + B_2 u - (a_2(q, \dot{q}, \epsilon\dot{H}) + A_2(q)H + B_2 u_s)$$

or,

$$\epsilon\ddot{\eta} = [a_2(q,\dot{q},\epsilon\dot{z}) - a_2(q,\dot{q},\epsilon\dot{H})] + A_2(q)\eta + B_2u_f \tag{28}$$

Substitute the value of a_2 and use fast time scale $\tau = \frac{t}{\sqrt{\epsilon}}$ with some manipulations we reach to [8]:

$$\epsilon\ddot{\eta} = A_2(q)\eta + B_2u_f \tag{29}$$

and in state space form:

$$\epsilon\begin{bmatrix}\dot{\eta}\\ \ddot{\eta}\end{bmatrix} = \begin{bmatrix}\emptyset & \epsilon I\\ A_2(q) & \emptyset\end{bmatrix}\begin{bmatrix}\eta\\ \dot{\eta}\end{bmatrix} + \begin{bmatrix}\emptyset\\ B_2\end{bmatrix}u_f \tag{30}$$

The flexible modes are not stable since the eigenvalues are on the imaginary axis. Hence, u_f must be designed such that the eigenvalues are shifted to the open left half plane in order to guarantee stability.

Theorem 1: The diagonal and positive definite matrices K_{pf} and K_{vf} exist such that the closed loop system including the subsystem 29 with the control effort $u_f = K_{pf}\eta + K_{vf}\dot{\eta}$ becomes globally asymptotically stable. (Proof in [8])

B. Control of Reduced Flexible Model

The reduced flexible model represents the effect of flexibility in the form of the flexible integral manifold. In this section a robust control algorithm is proposed for the system based on this model. In order to accurately derive a robust control law $u_s(q,\dot{q},\epsilon)$ for the system, manipulation of partial differential equation is necessary. To avoid complex manipulations, we propose deriving the robust control law $u_s(q,\dot{q},\epsilon)$ to any order of ϵ from the series expansion of the integral manifold to the same order of ϵ.

$$H(q,\dot{q},u_s,\epsilon) = H_0(q,\dot{q},u_s) + \epsilon H_1(q,\dot{q},u_s) + \cdots \tag{31}$$

and implement the controller $u_s(q,\dot{q},\epsilon)$ in the same form as:

$$u_s(q,\dot{q},\epsilon) = u_0(q,\dot{q}) + \epsilon u_1(q,\dot{q}) + \cdots \tag{32}$$

in which the functions $H_i(q,\dot{q},u_s), u_i(q,\dot{q}),\ i = 0,1,\cdots$ are calculated iteratively without need to solve the partial differential equations. It is important to note that as $\epsilon \to 0$, u_s tends to rigid control, and H tends to rigid integral manifold. By substitution of Equations 31 and 32 into manifold condition 22 we reach to:

$$\begin{aligned}\epsilon\ddot{H}_0(q,\dot{q},u_s) + \epsilon^2\ddot{H}_1(q,\dot{q},u_s) + \cdots \;=\;& a_2(q,\dot{q},\epsilon\dot{H}_0 + \epsilon^2\dot{H}_1 + \cdots) + A_2(q)(H_0 + \epsilon H_1 + \cdots) +\\ & B_2(u_0 + \epsilon u_1 + \cdots)\end{aligned} \tag{33}$$

The right hand side of Equation 33 can be expanded with respect to the powers of ϵ and by addition of equal powers of ϵ, a set of equations for $H_i, u_i,\ i = 0,1,\cdots$ in term of ϵ are resulted. The first order approximation of Equation 33 will result in:

$$\epsilon\ddot{H}_0(q,\dot{q},u_s) = a_2(q,\dot{q},\epsilon\dot{H}_0) + A_2(q)(H_0 + \epsilon H_1) + B_2(u_0 + \epsilon u_1) + O(\epsilon^2) \tag{34}$$

When $\epsilon = 0$ the equation relating u_0 to H_0 will be:

$$0 = a_{20} + A_2(q)H_0(q,\dot{q},u_0) + B_2u_0 \tag{35}$$

in which:

$$a_{20} = a_2(q, \dot{q}, 0) = J^{-1}D\dot{q} - J^{-1}T_F(q, \dot{q}) - M^{-1}(q)N(q, \dot{q}) \tag{36}$$

u_0 is designed using a robust design technique based on the rigid reduced order model ($\epsilon = 0$), and H_0 is calculated from:

$$H_0 = -A_2^{-1}(a_{20} + B_2 u_0) \tag{37}$$

The details of robust design technique is explained in the next section. Now, since u_0 and H_0 are known, from Equation 34, H_1 can be similarly calculated in terms of u_1, and the first order manifold H_1 can be substituted into the reduced flexible model (Equation 24). If higher order terms are neglected, the first order corrected model for the system is derived. In order to calculate H_1 from H_0 and u_0, let:

$$a_2(q, \dot{q}, \epsilon\dot{H}) = a_{20} + \epsilon\Delta a_2 + O(\epsilon^2)$$

in which a_{20} is given in Equation 36, and compare to Equation 13 we reach to:

$$\begin{cases} \Delta a_2 = -J^{-1}D\dot{H} \\ \Delta a_{20} = -J^{-1}D\dot{H}_0 \end{cases}$$

Hence,

$$\epsilon\ddot{H}_0 = a_{20} + A_2 H_0 + B_2 u_0 + \epsilon(\Delta a_{20} + A_2 H_1 + B_2 u_1) + O(\epsilon^2) \tag{38}$$

Compare Equation 38 to 35:

$$\ddot{H}_0 = \Delta a_{20} + A_2 H_1 + B_2 u_1 \tag{39}$$

Therefore,

$$H_1 = A_2^{-1}(\ddot{H}_0 - \Delta a_{20} - B_2 u_1) \tag{40}$$

To calculate u_1 refer to reduced flexible model 24 and approximate it to the first power of ϵ:

$$\ddot{q} = a_1(q, \dot{q}) + A_1(q)H_0 + \epsilon A_1(q)A_2^{-1}(\ddot{H}_0 - \Delta a_{20} - B_2 u_1)$$

By factoring the equal powers of ϵ we reach to:

$$u_1 = B_2^{-1}(\ddot{H}_0 - \Delta a_{20}) \tag{41}$$

The only condition on robust control design is that u_0 must be at least twice differentiable. Finally, the control law for slow subsystem has the form:

$$u_s = u_0 + \epsilon u_1 \tag{42}$$

In which u_1 is called the corrective term which is derived through this subsection and u_0 is the robust control based on the rigid model elaborated in the next section.

C. Robust PID Control for Rigid model

In this section we first propose a robust PID controller based on the rigid model of the system and then prove its robust stability with respect to the model uncertainties. Recall the rigid model of the system from Equation 15, choose a PID controller for u_0:

$$u_0 = K_V\dot{e} + K_P e + K_I \int_0^t e(s)ds = Kx \tag{43}$$

in which

$$\begin{cases} e = q_d - q \\ K = [K_I \quad K_P \quad K_V] \\ x = [\int_0^t e^T(s)ds \quad e^T \quad \dot{e}^T]^T \end{cases}$$

Similar to [4], [13] and [14], assume:

$$\underline{m}_t I \leq M_t(q) \leq \overline{m}_t I \tag{44}$$

and put some limits on:

$$\|N_t\| \leq \beta_0 + \beta_1\|L\| + \beta_2\|L\|^2 \quad ; \quad \|V_m\| \leq \beta_3 + \beta_4\|L\| \tag{45}$$

in which $\|.\|$ is the Euclidean norm and $L = [e^T \quad \dot{e}^T]$. Implement the control law u_0 in 15 to get:

$$\dot{x} = Ax + B\Delta A \tag{46}$$

where

$$A = \begin{bmatrix} \emptyset & I_n & \emptyset \\ \emptyset & \emptyset & I_n \\ -M_t^{-1}K_I & -M_t^{-1}K_P & -M_t^{-1}K_V \end{bmatrix} \quad B = \begin{bmatrix} \emptyset \\ \emptyset \\ M_t^{-1} \end{bmatrix}$$

$$\Delta A = N_t + M_t\ddot{q}_d \tag{47}$$

To analyze the system robust stability consider the following Lyapunov function:

$$V(x) = x^T P x = \frac{1}{2}[\alpha_2 \int_0^t e(s)ds + \alpha_1 e + \dot{e}]^T.M_t.[\alpha_2 \int_0^t e(s)ds + \alpha_1 e + \dot{e}] + w^T P_1 w \tag{48}$$

in which

$$w = \begin{bmatrix} \int_0^t e(s)ds \\ e \end{bmatrix} \quad P_1 = \frac{1}{2}\begin{bmatrix} \alpha_2 K_P + \alpha_1 K_I & \alpha_2 K_V + K_I \\ \alpha_2 K_V + K_I & \alpha_1 K_V + K_P \end{bmatrix}$$

Hence,

$$P = \frac{1}{2}\begin{bmatrix} \alpha_2 K_P + \alpha_1 K_I + \alpha_2^2 M_t & \alpha_2 K_V + K_I + \alpha_1\alpha_2 M_t & \alpha_2 M_t \\ \alpha_2 K_V + K_I + \alpha_1\alpha_2 M_t & \alpha_1 K_V + K_P + \alpha_1^2 M_t & \alpha_1 M_t \\ \alpha_2 M_t & \alpha_1 M_t & M_t \end{bmatrix}$$

Since M_t is a positive definite matrix, P is positive definite, if and only if, P_1 is positive definite. Now choose,

$$\begin{cases} K_P = k_P I \\ K_V = k_V I \\ K_I = k_I I \end{cases}$$

such that,

$$\begin{bmatrix} \alpha_2 k_P + \alpha_1 k_I & \alpha_2 k_V + k_I \\ \alpha_2 k_V + k_I & \alpha_1 k_V + k_P \end{bmatrix}$$

becomes positive definite. The following Lemma gives the conditions where V can become positive definite and upper and lower bounded.

Lemma 1: Assume the following inequalities hold:

$$\alpha_1 > 0 \quad \alpha_2 > 0 \quad \alpha_1 + \alpha_2 < 1$$
$$s_1 = \alpha_2(k_P - k_V) - (1-\alpha_1)k_I - \alpha_2(1+\alpha_1-\alpha_2)\overline{m}_t > 0$$
$$s_2 = k_P + (\alpha_1 - \alpha_2)k_V - k_I - \alpha_1(1+\alpha_2-\alpha_1)\overline{m}_t > 0$$

Then P is positive definite and satisfies the following inequality (Rayleigh-Ritz)[11]:

$$\underline{\lambda}(P)\|x\|^2 \leq V(x) \leq \overline{\lambda}(P)\|x\|^2 \tag{49}$$

in which,

$$\underline{\lambda}(P) = min\{\frac{1-\alpha_1-\alpha_2}{2}\underline{m}_t, \frac{s_1}{2}, \frac{s_2}{2}\}$$
$$\overline{\lambda}(P) = max\{\frac{1+\alpha_1+\alpha_2}{2}\overline{m}_t, \frac{s_3}{2}, \frac{s_4}{2}\}$$

and

$$s_3 = \alpha_2(k_P + k_v) + (1+\alpha_1)k_I + (1+\alpha_1+\alpha_2)\alpha_2\overline{m}_t$$
$$s_4 = \alpha_1\overline{m}_t(1+\alpha_1+\alpha_2) + (\alpha_1+\alpha_2)k_V + k_P + k_I$$

Proof is based on Gershgorin theorem and is similar to that in [13]. Now when P is positive definite then:

$$\dot{V}(x) = x^T(A^TP + PA + \dot{P})x + 2x^TPB\Delta A \tag{50}$$

$$\begin{aligned}\dot{V}(x) &= -x^TQx + \frac{1}{2}x^T\begin{bmatrix}\alpha_2 I\\ \alpha_1 I\\ I\end{bmatrix}\dot{M}_t\,[\,\alpha_2 I \quad \alpha_1 I \quad I\,]\,x + x^T\begin{bmatrix}\alpha_2 I\\ \alpha_1 I\\ I\end{bmatrix}\Delta A \\ &+ \frac{1}{2}x^T\begin{bmatrix}\emptyset & \alpha_2^2 I & \alpha_1\alpha_2 I\\ \alpha_2 I & 2\alpha_1\alpha_2 I & (\alpha_1^2+\alpha_2)I\\ \alpha_1\alpha_2 I & (\alpha_1^2+\alpha_2)I & \alpha_1 I\end{bmatrix}\begin{bmatrix}M_t & \emptyset & \emptyset\\ \emptyset & M_t & \emptyset\\ \emptyset & \emptyset & M_t\end{bmatrix}x\end{aligned} \tag{51}$$

refer to [12]

$$y^T\dot{M}_t y = 2y^TV_m y$$

with some manipulations we can show [8]:

$$\dot{V}(x) \leq -\gamma\|x\|^2 + \lambda_1\|V_m\|\|x\|^2 + \lambda_2\overline{m}_t\|x\|^2 + \alpha_2^{-1}\lambda_1\|x\|\|\Delta A\|$$
$$\dot{V}(x) \leq \|x\|(\xi_0 - \xi_1\|x\| + \xi_2\|x\|^2) \tag{52}$$

and

$$\gamma = min\{\alpha_2 k_I, \alpha_1 k_P - \alpha_2 k_V - k_I, k_V\}$$

Now considering Equations 45, 47 and 52 and $\|L\| \leq \|x\|$ then,

$$\begin{aligned}\xi_0 &= \alpha_2^{-1}\lambda_1\beta_0 + \alpha_2^{-1}\lambda_1\lambda_3\overline{m}_t\\ \xi_1 &= \gamma - \lambda_1\beta_3 - \lambda_2\overline{m}_t - \alpha_2^{-1}\lambda_1\beta_1\\ \xi_2 &= \lambda_1\beta_4 + \alpha_2^{-1}\lambda_1\beta_2\end{aligned}$$

in which

$$\begin{cases} \lambda_1 = \lambda_{max}(R_1) \\ \lambda_2 = \lambda_{max}(R_2) \\ \lambda_3 = sup\|\ddot{q}_d\| \end{cases}$$

and $\lambda_{Min}, \lambda_{Max}$ are the least and largest eigenvalues, respectively, and

$$R_1 = \begin{bmatrix} \alpha_2^2 I & \alpha_1\alpha_2 I & \alpha_2 I \\ \alpha_1\alpha_2 I & \alpha_1^2 I & \alpha_1 I \\ \alpha_2 I & \alpha_1 I & I \end{bmatrix}$$

$$R_2 = \frac{1}{2}\begin{bmatrix} \emptyset & \alpha_2^2 I & \alpha_1\alpha_2 I \\ \alpha_2 I & 2\alpha_1\alpha_2 I & (\alpha_1^2+\alpha_2)I \\ \alpha_1\alpha_2 I & (\alpha_1^2+\alpha_2)I & \alpha_1 I \end{bmatrix}$$

According to the result obtained so far, we can proof the stability of the error system based on the following theorem.

Theorem 2: The error system 46 is stable of the form of UUB, if ξ_1 is chosen large enough.

Proof: According to Equations 52 and 49 and the Lemma 3.5 from [12], if the following condition hold, the system is UUB stable with respect to $B(0, d)$, where

$$d = \frac{2\xi_0}{\xi_1 + \sqrt{\xi_1^2 - 4\xi_0\xi_2}}\sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)}}$$

The conditions are:

$$\begin{aligned} \xi_1 &> 2\sqrt{\xi_0\xi_2} \\ \xi_1^2 + \xi_1\sqrt{\xi_1^2 - 4\xi_0\xi_2} &> 2\xi_0\xi_2\Big(1 + \sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)}}\Big) \\ \xi_1 + \sqrt{\xi_1^2 - 4\xi_0\xi_2} &> 2\xi_2\|x_0\|\sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)}} \end{aligned}$$

These conditions can be simply met by making ξ_1 and the control gains K_P, K_v, and K_I large enough. ■

V. Stability Analysis of the Complete Closed-loop System

The stability of the fast, and slow subsystems are separately analyzed in previous sections. However, the stability of the complete closed-loop system may not be guaranteed through these separate analysis [10]. In this section the stability of the complete system is analyzed. Recall the dynamic equations of the FJR Equation 11. The integral manifold and the control effort are chosen as:

$$\eta = z - H$$

$$H = H_0 + \epsilon H_1$$

$$u = u_s + u_f = u_0 + \epsilon u_1 + u_f$$

Combine these equations to Equation 11, 40, 37 and 43, and consider, $x = [\int_0^t e(s)^T ds \quad e^T \quad \dot{e}^T]^T$ and $y = [\eta^T \quad \dot{\eta}^T]^T$ then,

$$\dot{x} = Ax + B\Delta A + C\,[I \quad \emptyset]\,y \tag{53}$$

$$\epsilon\dot{y} = \tilde{A}y \tag{54}$$

in which,

$$A = \begin{bmatrix} \emptyset & I & \emptyset \\ \emptyset & \emptyset & I \\ -M_t^{-1}K_I & -M_t^{-1}K_P & -M_t^{-1}K_V \end{bmatrix} \; ; \; B = \begin{bmatrix} \emptyset \\ \emptyset \\ M_t^{-1} \end{bmatrix}$$

$$\Delta A = N_t + M_t \ddot{q}_d$$

$$C = \begin{bmatrix} \emptyset \\ \emptyset \\ -A_1 \end{bmatrix} \quad ; \quad \tilde{A} = \begin{bmatrix} \emptyset & \epsilon I \\ A_2 + B_2 K_{pf} & -\epsilon J^{-1} D + B_2 K_{vf} \end{bmatrix}$$

Theorem 3: There exist diagonal and positive definite matrices K_{pf} and K_{vf} such that the closed loop system 54 becomes globally asymptotically stable.

Proof: Substitute A_2, B_2 from Equations 14 into 54, and define:

$$M^{-1} + J^{-1} + J^{-1}K_{pf} = U$$
$$\epsilon J^{-1} D + J^{-1} K_{vf} = G$$

Where U and G are both positive definite, since M, J, K_{pf} and K_{vf} are all positive definite, hence,

$$\epsilon \begin{bmatrix} \dot{\eta} \\ \ddot{\eta} \end{bmatrix} = \begin{bmatrix} \emptyset & \epsilon I \\ -U & -G \end{bmatrix} \begin{bmatrix} \eta \\ \dot{\eta} \end{bmatrix}$$

Consider the following Lyapunov function:

$$V_F = y^T S y$$

in which $y = [\eta \;\; \dot{\eta}]^T$ and,

$$S = \frac{1}{2} \begin{bmatrix} \frac{2}{\epsilon} I & G^{-1} \\ G^{-1} & U^{-1} \end{bmatrix}$$

In order to have positive definite S, according to Shur Complement we must have:

$$\begin{cases} \frac{2}{\epsilon} I > 0 \\ U^{-1} - G^{-1}(\frac{2}{\epsilon} I)^{-1} G^{-1} > 0 \Longrightarrow U^{-1} - \frac{\epsilon}{2} G^{-2} > 0 \end{cases} \tag{55}$$

Now since U^{-1}, G^{-2} are positive definite in order to satisfy 55 the following condition must be met:

$$\epsilon < \frac{2\lambda_{min}(U^{-1})}{\lambda_{min}(G^{-2})}$$

in which λ_{min} is the smallest eigenvalue. Differentiate V_F along trajectories of 54

$$\dot{V}_F = \dot{y}^T S y + y^T S \dot{y} + y^T \dot{S} y = -\frac{1}{\epsilon} \eta^T G^{-1} U \eta - \dot{\eta}^T [\frac{1}{\epsilon} U^{-1} G - \frac{1}{2}(G^{-1} + (U^{-1})')] \dot{\eta} < 0$$

Since G is diagonal and positive definite, and limited $q, \dot{q}$ will limit $(U^{-1})'$, then by choosing appropriate values for K_{vf}, K_{pf}, $\dot{V}_F$ becomes negative definite and it can be written as:

$$\dot{V}_F = -y^T W y$$

in which

$$W = \frac{1}{2}\begin{bmatrix} \frac{1}{\epsilon}(G^{-1}U + UG^{-1}) & \emptyset \\ \emptyset & \frac{1}{\epsilon}(U^{-1}G + GU^{-1}) - (U^{-1})' - G^{-1} \end{bmatrix}$$

■

Theorem 4: The closed–loop system of Equations 53 and 54 is UUB stable if K_{pf}, K_{vf}, and ξ_1 are chosen large enough.

Proof: Consider the following composite Lyapunov function:

$$V(x, y) = x^T Px + y^T Sy \tag{56}$$

where $x^T Px$ is the Lyapunov function candidated for slow subsystem, and $y^T Sy$ is the Lyapunov function of fast subsystem 54. Therefore, from Rayleigh–Ritz inequality:

$$\begin{aligned} \underline{\lambda}(P)\|x\|^2 \le x^T Px \le \overline{\lambda}(P)\|x\|^2 \\ \underline{\lambda}(S)\|y\|^2 \le y^T Sy \le \overline{\lambda}(S)\|y\|^2 \end{aligned}$$

in which $\overline{\lambda}$ and $\underline{\lambda}$ are the largest and smallest eigenvalue respectively. Adding the above inequalities:

$$\underline{\lambda}(S)\|y\|^2 + \underline{\lambda}(P)\|x\|^2 \le V(x, y) \le \overline{\lambda}(S)\|y\|^2 + \overline{\lambda}(P)\|x\|^2$$

Define,

$$Z_t = [\,\|x\| \quad \|y\|\,]^T \tag{57}$$

then,

$$\begin{aligned} [\,\|x\| \quad \|y\|\,]\begin{bmatrix} \underline{\lambda}(P) & 0 \\ 0 & \underline{\lambda}(S) \end{bmatrix}\begin{bmatrix} \|x\| \\ \|y\| \end{bmatrix} \le V(x, y) \\ \le [\,\|x\| \quad \|y\|\,]\begin{bmatrix} \overline{\lambda}(P) & 0 \\ 0 & \overline{\lambda}(S) \end{bmatrix}\begin{bmatrix} \|x\| \\ \|y\| \end{bmatrix} \end{aligned}$$

Again apply Rayleigh–Ritz inequality:

$$\underline{\lambda}\|Z_t\| \le V(Z_t) \le \overline{\lambda}\|Z_t\| \tag{58}$$

where

$$\begin{aligned} \underline{\lambda} = Min\{\underline{\lambda}(P), \underline{\lambda}(S)\} \\ \overline{\lambda} = Max\{\overline{\lambda}(P), \overline{\lambda}(S)\} \end{aligned}$$

Now differentiate 56 along trajectories of 53 and 54,

$$\begin{aligned} \dot{V} = 2x^T P\dot{x} + x^T \dot{P}x + 2y^T S\dot{y} + y^T \dot{S}y = 2x^T PC[I \quad \emptyset]y + \\ [2x^T P(Ax + B\Delta A) + x^T \dot{P}x] + 2y^T S\dot{y} + y^T \dot{S}y \end{aligned}$$

and consider Equations 50 and 52, and define $\gamma_1 = \lambda_{max}(M^{-1})$ As it is shown in Theorem (3),

$$2y^T S\dot{y} + y^T \dot{S}y \le -\lambda_{min}(W)\|y\|^2$$

Hence

$$\dot{V} \leq -[\,\|x\| \quad \|y\|\,]\begin{bmatrix}\xi_1 & -\gamma_1\overline{\lambda}(P)\\ -\gamma_1\overline{\lambda}(P) & \lambda_{min}(W)\end{bmatrix}\begin{bmatrix}\|x\|\\ \|y\|\end{bmatrix} + \xi_0\|x\| + \xi_2\|x\|^3$$

And according to 57

$$\dot{V} \leq -Z_t^T R Z_t + \xi_0\|Z_t\| + \xi_2\|Z_t\|^3$$

where,

$$R = \begin{bmatrix}\xi_1 & -\gamma_1\overline{\lambda}(P)\\ -\gamma_1\overline{\lambda}(P) & \lambda_{min}(W)\end{bmatrix}$$

In order to have positive definite R

$$\Delta_1 : \xi_1\lambda_{min}(W) - \gamma_1^2\overline{\lambda}^2(P) > 0$$

or,

$$\lambda_{min}(W) > \frac{\gamma_1^2\overline{\lambda}^2(P)}{\xi_1} \tag{59}$$

Condition 59 is met by choosing appropriate K_{pf} and K_{vf} for fast subsystem, hence,

$$\dot{V} \leq \|Z_t\|(\xi_0 - \lambda_{min}(R)\|Z_t + \xi_2\|Z_t\|^2) \tag{60}$$

Now, according to Equations 60 and 58 and the Lemma 3.5 of [12], if these conditions are met then the closed–loop system is UUB stable with respect to $Y(0, d')$, where

$$d' = \frac{2\xi_0}{\lambda_{min}(R) + \sqrt{\lambda_{min}^2(R) - 4\xi_0\xi_2}}\sqrt{\frac{\overline{\lambda}}{\underline{\lambda}}}$$

and the stability conditions are:

$$\lambda_{min}(R) > 2\sqrt{\xi_0\xi_2}$$

$$\lambda_{min}^2(R) + \lambda_{min}(R)\sqrt{\lambda_{min}^2(R) - 4\xi_0\xi_2} > 2\xi_0\xi_2(1 + \sqrt{\tfrac{\overline{\lambda}}{\underline{\lambda}}})$$

$$\lambda_{min}(R) + \sqrt{\lambda_{min}^2(R) - 4\xi_0\xi_2} > 2\xi_2\|Z_{t0}\|\sqrt{\tfrac{\overline{\lambda}}{\underline{\lambda}}}$$

■

These conditions are simply met by increasing $\lambda_{min}(R)$, through appropriate choice of large ξ_1, and $\lambda_{min}(W)$. ξ_1 is a function of the robust PID gains K_p, K_I and K_V, and $\lambda_{min}(W)$ are affected by the fast subsystem gains K_{pf} and K_{vf}. Therefore, by the choice of the controller gains such that the above conditions are met the robust stability of the closed–loop system is guaranteed.

VI. Simulations

In order to verify the effectiveness of the algorithm a simulation study has been forwarded next. In the following simulation study, the results of the closed loop performance of two flexible joint manipulators examined in the literature is compared to that of the proposed control algorithm. First a single joint manipulator examined in detail by spong et al [17], has been simulated, and the closed loop performances are compared. Then, the two link manipulator which has been studied by Al-Ashoor et al [1], is examined in detail and a robust PID controller is designed for each joint. Moreover, the closed loop performance of this system is presented. The simulation results show the effectiveness of the proposed algorithm, despite the simplicity of its structure, and the convenience of its online implementation.

A. Single Link Flexible Joint Manipulator

Consider the single link flexible joint manipulator introduced in [17].The dynamic equation of motion of this system is as follows:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-MgL}{I}\sin(x_1) - \frac{K}{I}(x_1 - x_3) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{K}{J}(x_1 - x_3) + \frac{1}{J}u \end{aligned} \tag{61}$$

in which $x_1 = q_1$ and $x_2 = q_2$. In the limit of $k \to \infty$ the rigid model of the system is given by:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-MgL}{I+J}\sin(x_1) - \frac{1}{I+J}u \end{aligned} \tag{62}$$

in which $x_1 = q_1 = q_2$. By choosing $q_1 = q$ and $z = K(q_1 - q_2$ as the alastic force, the model of the system can be rewritten in a singular perturbation form:

$$\begin{aligned} \ddot{q} &= \frac{-MgL}{I}\sin(q) - \frac{1}{I}z \\ \epsilon\ddot{z} &= \frac{-MgL}{I}\sin(q) - (\frac{1}{I} + \frac{1}{J})z - \frac{1}{J}u \end{aligned} \tag{63}$$

in which $\epsilon = \frac{1}{K}$.

Spong has proposed a composite control law for this system, in which there exists two control components corresponding to the fast and slow dynamics. The slow dynamic component is composed of a control law based on the rigid model of the system in addition to a corrective term, which is a feedback linearization algorithm based on the rigid model of the system [17]. According to the rigid model of the system given is Equation 62 the feedback linearization control signal can be chosen as:

$$u_0 = (I + J)v + Mgl\sin(x_1) \tag{64}$$

in which v is linear component of it, and can be given as:

$$v = \dot{x}_2^d - a_1(x_1 - x_1^d) - a_2(x_2 - x_2^d) \tag{65}$$

In order to derive the corrective term, the integral manifold and the control law are expanded as follows:

$$\begin{aligned} H &= H_o + \epsilon H_1 + O(\epsilon^2) \\ u_s &= u_o + \epsilon u_1 + O(\epsilon^2) \end{aligned}$$

Substitute these relation into Equation 63 and equating the corresponding terms, we have:

$$H_o = \frac{-MgLJ}{I+J}\sin(q) - \frac{I}{I+J}u_o \tag{66}$$

and similarly,

$$u_1 = \ddot{H}_o \tag{67}$$

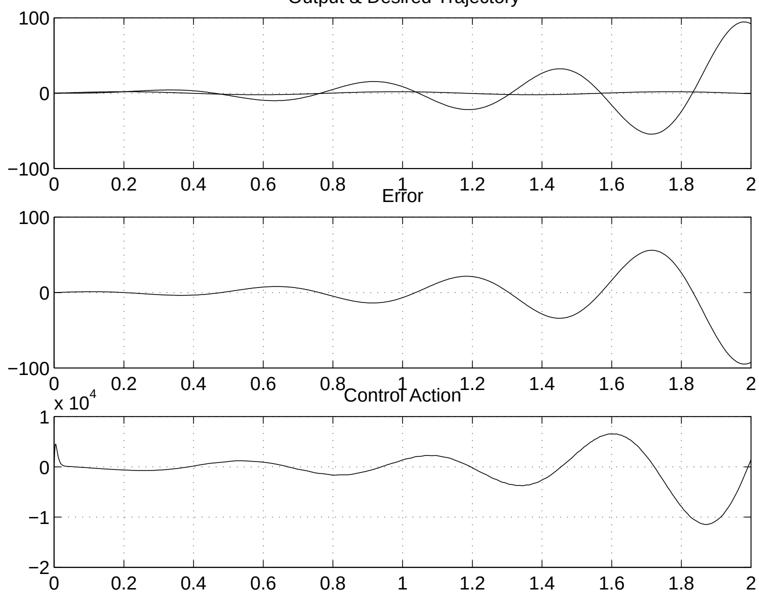

Fig. 2. Instability of closed loop system by applying only rigid controller term u_o; Spong algorithm

Choose

$$u_f = 1\eta + 1\dot{\eta} \tag{68}$$

in which η corresponds to the variation of z from the manifold H. Hence, the composite control law is given by:

$$u = u_s + u_f = U_o + \epsilon u_1 + u_f \tag{69}$$

in which u_0, u_1, and u_f are evaluated in Equation 64, Equation 67 and Equation 68, respectively.

As it is illustrated in the Figure 2, the closed loop system became unstable, provided that only the corresponding rigid control effort u_0 is applied on the system. However, as illustrated in Figure 3 the system becomes stable and the desired trajectory $q_d = \sin(8t)$ is well tracked, implementing the proposed composite control on the nominal model of the system. However, this

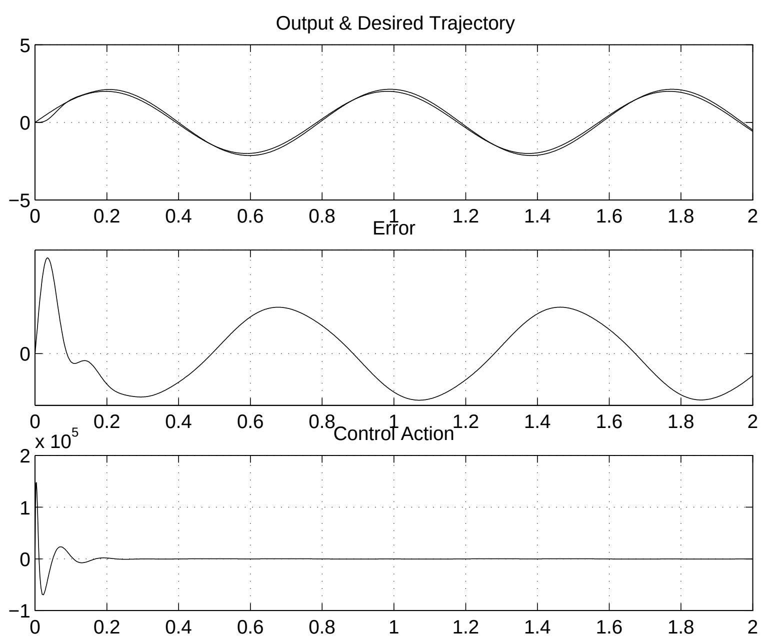

Fig. 3. Tracking performance of the closed loop system for nominal model; Spong algorithm

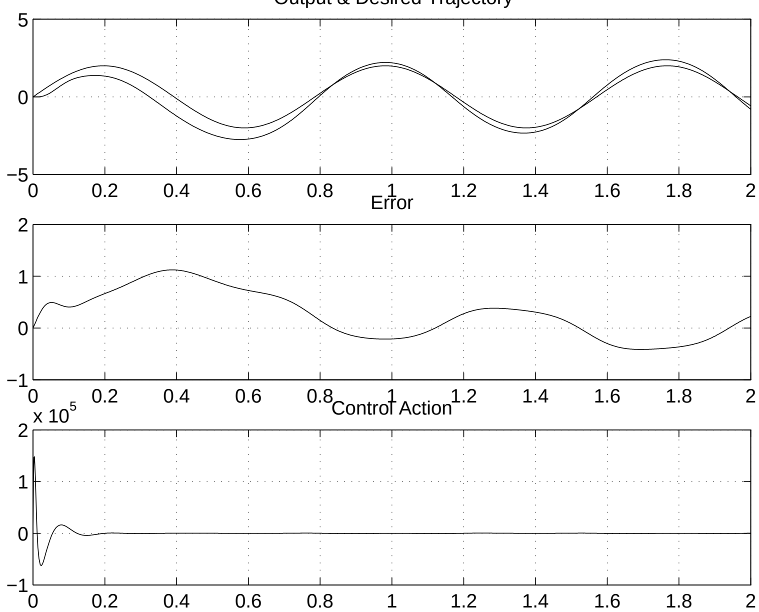

Fig. 4. Poor tracking performance of the closed loop system for perturbed model; Spong algorithm

algorithm is not robust to the model parameter variations. As illustrated in Figure 4 the tracking performance is getting quite poor for the maximum perterbation values for he parameters I, J, M, and L.

For the sake of comparison, the proposed robust PID controller may be now applied on the same system. The porpsed control law is composed of three terms as given in Equation 69, in which the rigid control law is a PID controller whose coefficients satisfies the robust stability conditions elaborated in Theorem (4) as following:

$$u_o = 200\dot{e} + 500e + 100 \int_0^t e(s)ds.$$

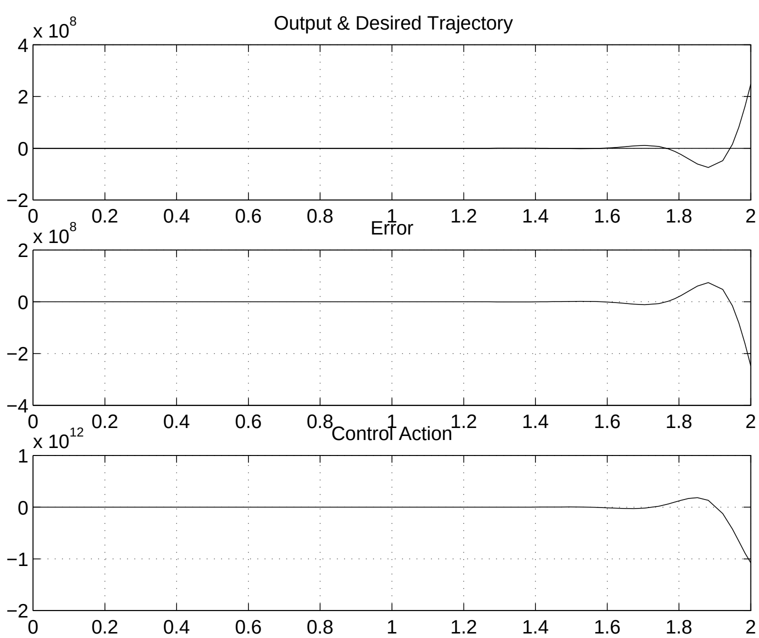

Fig. 5. Instability of closed loop system by applying only rigid controller term u_o; Proposed algorithm

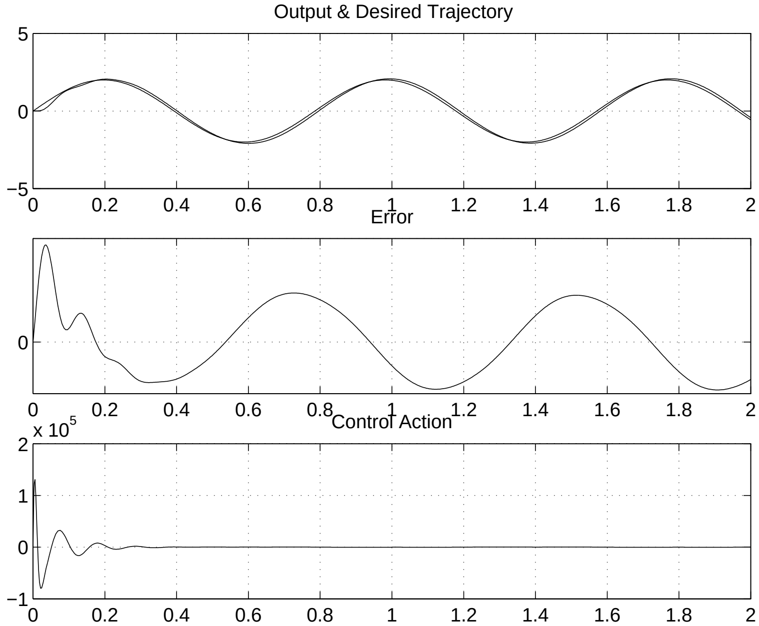

Fig. 6. Tracking performance of the closed loop system for nominal model; Proposed algorithm

The integral manifold would be:

$$H_o = -4.9\sin(q) - \frac{1}{2}u_o$$

and the corrective term corresponds to

$$u_1 = \ddot{H}_o.$$

The fast control law is a simple PD controller satisfying the robust stability conditions such as:

$$u_f = 5\eta + 5\dot{\eta}$$

in which η indicates the variation of z from the integral manifold H.

It is observed as before, that if only the rigid term of the composite control law is implemented on the closed loop system, the system becomes unstable as illustrated in Figure 5. However, By implementation of the complete proposed control law, not only the system is well tracking the desired trajectory for the nominal parameters of the model (refigfig:spong6), but also the robust stability and tracking performance of the system with maximum variation in its model parameters are preserved (Figure 7).

The simulation results show clearly the effectiveness of the proposed control algorithm to robustly stabilize the system, while achieving robust performance. The superiority of our proposed algorithm compared to Spong algorithm is its robustness to the system model variations, and the simplicity of its implementation. To quantitatively compare the tracking errors obtained by these methods, note that the two–norm of the tracking error in Spong algorithm is 20.73, while its infinity–norm is about 1,123. By our proposed method these values are reduce to 2.93, and 0.447, respectively, despite the similarity in the norm of the actuator efforts. Hence, the proposed algorithm is not only robust to the model variation, and quite simpler in structure, but also improves the tracking performance quite significantly.

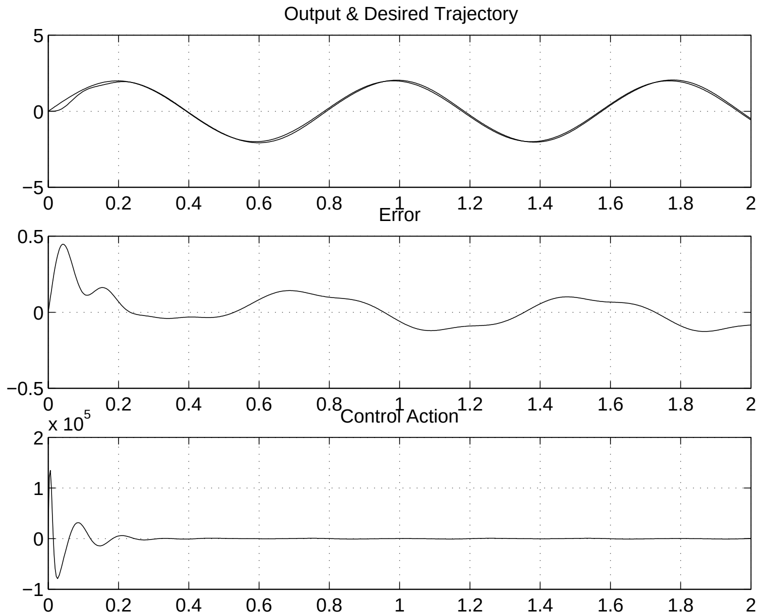

Fig. 7. Suitable tracking performance of the closed loop system for perturbed model; Proposed algorithm

B. Multiple Link Flexible Joint Manipulator

Consider the two link Flexible Joint manipulator illustrated in Figure 1. In this manipulator Joint flexibility is modeled with a linear torsional spring with stiffness k. The equation of motion of this system is as follows [1]:

$$\begin{aligned} m_{11}\ddot{\theta}_1 + m_{12}\ddot{\theta}_2 + C_{21}\dot{\theta}_2^{\,2} + G_1 + k_1(\theta_1 - \phi_1) &= 0 \\ m_{21}\ddot{\theta}_1 + m_{22}\ddot{\theta}_2 + C_{12}\dot{\theta}_1^{\,2} + G_2 + k_2(\theta_2 - \phi_2) &= 0 \\ N_1^2 J_{m1}\ddot{\phi}_1 - k_1(\theta_1 - \phi_1) = u_1 & \\ N_1^2 J_{m2}\ddot{\phi}_2 - k_2(\theta_2 - \phi_2) = u_2 & \end{aligned} \tag{70}$$

in which m_{ij} are the elements of the following mass matrix.

$$M(\theta, \phi) = \begin{bmatrix} m_1 l_{c1}^2 + m_2 l_1^2 + J_{l1} & m_2 l_1 l_{c2}\cos(\phi_1 - \theta_1) \\ m_2 l_1 l_{c2}\cos(\phi_1 - \theta_1) & m_2 l_{c2}^2 + J_{l2} \end{bmatrix} \tag{71}$$

m_i, J_{li} are the mass and the moment of inertia of the i'th link, while l_i, L_{ci} are the link length and the distance of the center of mass of i'th link to its joint, respectively. The other terms of Equation 71 are given as follows:

$$\begin{aligned} C_{21} = -m_2 l_1 l_{c2}\sin(\phi_1 - \theta_1) \quad &, \quad G_1 = (m_1 l_{c1} + m_2 l_1) g\cos(\theta_1) \\ C_{12} = m_2 l_1 l_{c2}\sin(\phi_1 - \theta_1) \quad &, \quad G_2 = m_2 l_{c2} g\cos(\theta_2) \end{aligned} \tag{72}$$

in which g is the gravity constant, k_i is the stiffness of i'th spring, J_i is the moment of inertia of i'th link and N_i is the i'th gearbox ratio. The numerical parameters used for simulations are as following [1]:

$$\begin{gathered} m_1 = m_2 = 1;\ J_{l1} = J_{l2} = 1;\ k_1 = k_1 = 100 \\ N_1^2 J_{m1} = N_2^2 J_{m2} = 1;\ l_1 = l_2 = 1;\ l_{c1} = l_{c2} = 0.5 \end{gathered}$$

Borrowing this system from [1], our proposed algorithm is applied to the system for comparison of the results. The equation of motion of the system can be reformulated in the standard form of singular perturbation, using $\epsilon = \frac{1}{k_1} = \frac{1}{k_2} = \frac{1}{k} = 0.01$ as the singular perturbation parameter. defining two new state variables $z_1 = k(\theta_1 - \phi_1), z_2 = k(\theta_2 - \phi_2)$ as the elastic torques in the compliant elements, then:

$$\begin{bmatrix} 2.25 & 0.5\cos(\epsilon z_1) \\ 0.5\cos(\epsilon z_1) & 1.25 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 14.7\cos(\theta_1) + 0.5\dot{\theta}_2^2 \sin(\epsilon z_1) \\ 4.9\cos(\theta_2) - 0.5\dot{\theta}_1^2 \sin(\epsilon z_1) \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (73)$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} - \epsilon \begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} - \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (74)$$

The corresponding rigid model when $k \to \infty$ will be:

$$\begin{bmatrix} 3.25 & 0.5 \\ 0.5 & 2.25 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 14.7\cos(\theta_1) \\ 4.9\cos(\theta_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (75)$$

As elaborated before the integral manifold for corresponding system can be defined as:

$$Z_1 = H_1(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, u_1, u_2, \epsilon) \;\; ; \;\; Z_2 = H_2(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, u_1, u_2, \epsilon) \quad (76)$$

in which H_1, H_2 satisfy the manifold condition. Expand the manifolds up to first degree:

$$H_1 = H_1^o + \epsilon H_1^1 + O(\epsilon^2) \;\; ; \;\; H_2 = H_2^o + \epsilon H_2^1 + O(\epsilon^2) \quad (77)$$

and expand the corresponding control efforts as:

$$u_{1s} = u_1^o + \epsilon u_1^1 + O(\epsilon^2) \;\; ; \;\; u_{2s} = u_2^o + \epsilon u_2^1 + O(\epsilon^2) \quad (78)$$

Hence the reduced order first order model is evaluated as following:

$$\begin{bmatrix} 2.25 & 0.5\cos(\epsilon H_1^o) \\ 0.5\cos(\epsilon H_1^o) & 1.25 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 14.7\cos(\theta_1) + 0.5\dot{\theta}_2^2 \sin(\epsilon H_1^o) \\ 4.9\cos(\theta_2) - 0.5\dot{\theta}_1^2 \sin(\epsilon H_1^o) \end{bmatrix} + \begin{bmatrix} H_1^o + \epsilon H_1^1 \\ H_2^o + \epsilon H_2^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (79)$$

In order to evaluate the fast dynamics caused by the joint flexibility, the normalized time variable $\tau = \frac{t}{\sqrt{\epsilon}}$ is used. Hence,

$$\begin{aligned} \frac{d^2\eta_1}{d\tau^2} &= \epsilon \frac{d^2\eta_1}{dt^2} = -\eta_1 - u_1^f(\eta_1, \eta_2) \\ \frac{d^2\eta_2}{d\tau^2} &= \epsilon \frac{d^2\eta_2}{dt^2} = -\eta_2 - u_2^f(\eta_1, \eta_2) \\ \eta_1 &= Z_1 - H_1^o \;\; ; \;\; \eta_2 = Z_2 - H_2^o \end{aligned} \quad (80)$$

in which H_1^o, H_2^o can be evaluated simply by replacing $\epsilon = 0$ in Equation 74.

$$H_1^o = \ddot{\theta}_1 - u_1^o \;\; ; \;\; H_2^o = \ddot{\theta}_2 - u_2^o \quad (81)$$

In order to evaluate the integral manifold, and the control law for this system, Equation 74 is used, substituting $Z_i = H_i$ and equating up to first order term with respect to ϵ. This concludes to,

$$H_1^1 = -\ddot{H}_1^o - u_1^1 \;\; ; \;\; H_2^1 = -\ddot{H}_2^o - u_2^1 \quad (82)$$

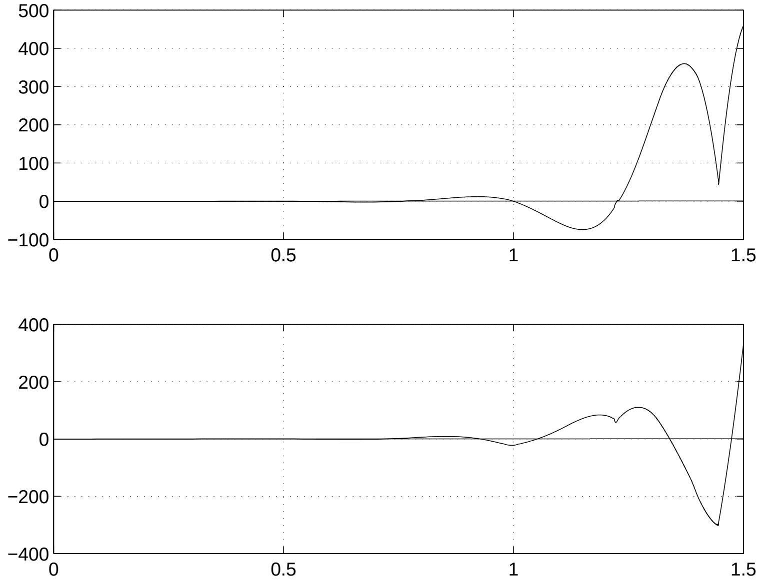

Fig. 8. Instability of closed loop system by applying only rigid controller term u_o; Proposed algorithm

With expanding Equation 81 to the first order of ϵ we have:

$$H_1^1 = -0.5\dot{\theta}_2^2 \, H_1^o \;\; ; \;\; H_2^1 = -0.5\dot{\theta}_1^2 \, H_1^o \tag{83}$$

And from Equation 82 we get:

$$u_1^1 = -0.5\dot{\theta}_2^2 \, H_1^o - \ddot{H}_1^o \;\; ; \;\; u_2^1 = -0.5\dot{\theta}_1^2 \, H_1^o - \ddot{H}_2^o \tag{84}$$

Finally, the slow part of the control law will be calculated from:

$$u_{1s} = u_1^o + \epsilon u_1^1 \;\; ; \;\; u_{2s} = u_2^o + \epsilon u_2^1 \tag{85}$$

The u_1^o, u_2^o are the rigid part of the control law and as elaborated before is robustly designed as a PID controller. In here we design the PID gains as following which satisfies the robust conditions:

$$\begin{aligned} u_1^o &= 500e + 50\dot{e} + 50 \int_0^t e(s)ds \\ u_2^o &= 200e + 50\dot{e} + 50 \int_0^t e(s)ds \end{aligned} \tag{86}$$

The fast control law is also designed as a PD controller as:

$$u_{1f} = \dot{\eta}_1 + \eta_1 \;\; ; \;\; u_{2f} = \dot{\eta}_2 + \eta_2 \tag{87}$$

Finally the control law is composed from the east and slow parts:

$$u_1 = u_{1s} + u_{1f} \;\; ; \;\; u_2 = u_{2s} + u_{2f} \tag{88}$$

B.1 Simulation Results

To have simulation results compared to [1], the reference signal is considered as:

$$\theta_i = 1.57 + 7.8539e^{-t} - 9.428e^{-t/1.2} \qquad i = 1, 2 \tag{89}$$

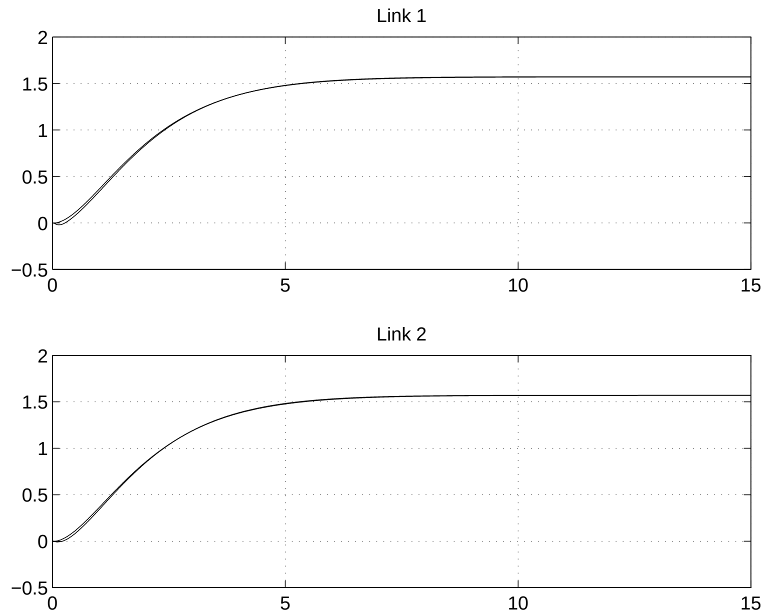

Fig. 9. Tracking performance of the closed loop system for nominal model; Proposed algorithm

in which the joint angles reach to a final value of $\theta_i = \frac{\pi}{2}$ from zero initial state. As shown in Figure 8, the system reaches to instability if the rigid control is applied to the system. The main reason for stability is the divergence of its fast dynamics.

Figure 9 illustrates the response of the system to our proposed composite control law. The system becomes stable, and the tracking performance is quite desirable, despite the limited control effort guaranteed with a adding a saturation block in the simulation (Figure 10). In order to analyze the robustness of the response, the system parameters are varied 50%. Figures 11 and 12 illustrate the robustness of the performance, and stability to the model variations.

In order to compare the effectiveness of our proposed control law, the simulation results are compared to the results presented in [1]. Al-Ashoor et al have used a robust–adaptive control law

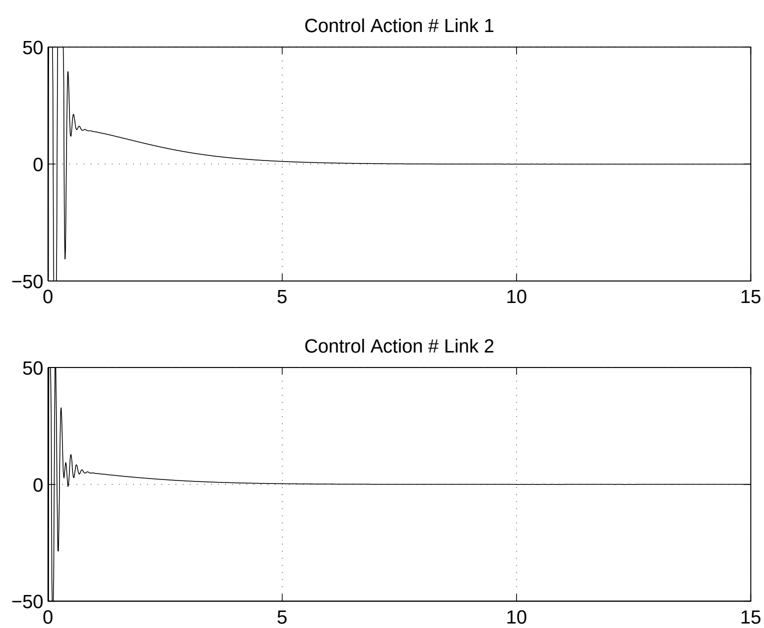

Fig. 10. Control effort for the closed loop system and nominal model; Proposed algorithm

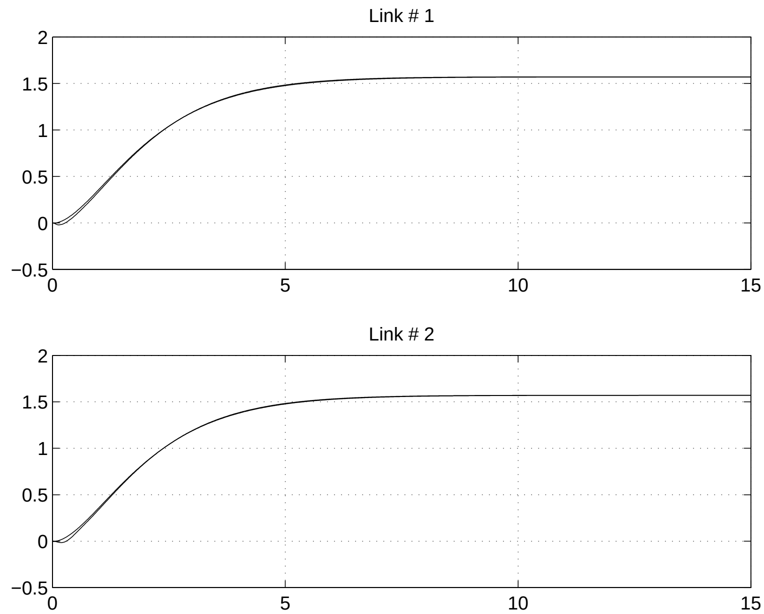

Fig. 11. Tracking performance of the closed loop system for Perturbed model; Proposed algorithm

in addition to the composite law we introduced in this paper. By this means, in addition to the corrective adaptive term used based on the integral manifold, another term is used for robustness of the performance against the modeling uncertainties. Figure 13 illustrates the results obtained for the reference signal introduced in Equation 89 in [1]. This figure illustrates the tracking performance despite the bounded control effort illustrated in Figure 14. Comparing these results to that obtained with our proposed control law (Figure 9 and Figure 10) it is clear that, despite the simplicity of our proposed control law the results are quite similar. Hence our proposed algorithm results into a much simpler implementation effort without loss of performance. The only limitation exists in our proposed law compared to that in [1], is the amplitude of the control law in the initial time of the simulation. The adaptive law have smaller control effort in the beginning of the simulation, due to the adaptive nature of the algorithm, and using the

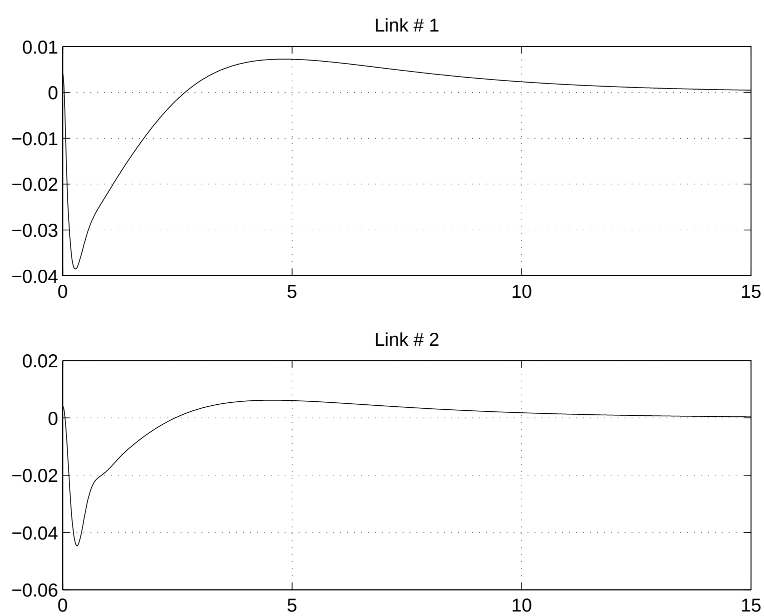

Fig. 12. Tracking Error for the closed loop system and perturbed model; Proposed algorithm

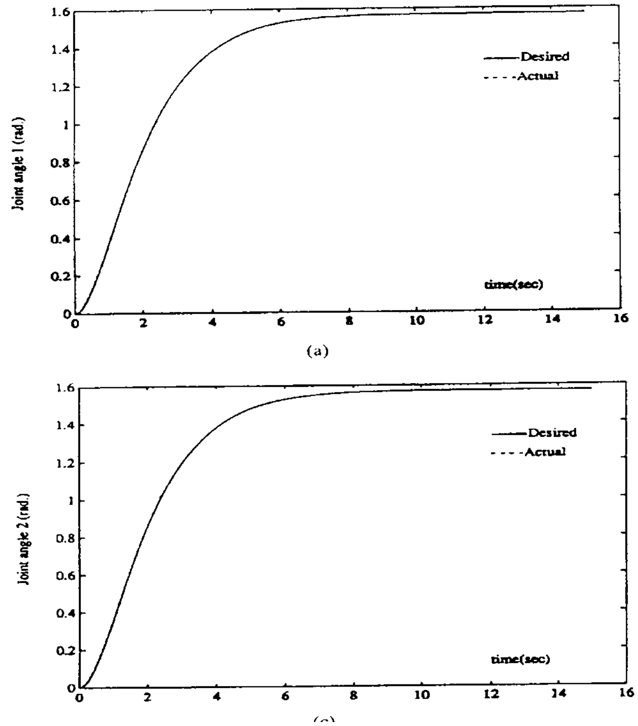

Fig. 13. Tracking performance of the closed loop system for nominal model; Al-Ashoor et al algorithm

information of the identified model of the system in the control law. This issue is under current investigation, and promising results are obtained by a $\mathcal{H}_\infty$ –based robust performance synthesis for PID design, in which the control effort can be limited to desirable bounds, [19].

VII. Conclusions

In this paper the control of flexible joint manipulators is examined in detail. First the model of N–axis robot manipulators are given and reformulated in the form of singular perturbations. Rigid and flexible integral manifolds are defined for the singularly perturbed model of the system, and fast and slow subsystems are partitioned by them. In order to achieve the required performance a composite control algorithm is proposed, consisting of corresponding control law for fast and slow subsystems. A simple PD control is proposed for the fast subsystem, and it is proven that the fast subsystem becomes asymptotically stable and the flexible manifold is invariant. The slow subsystem itself is controlled through a robust PID controller designed based

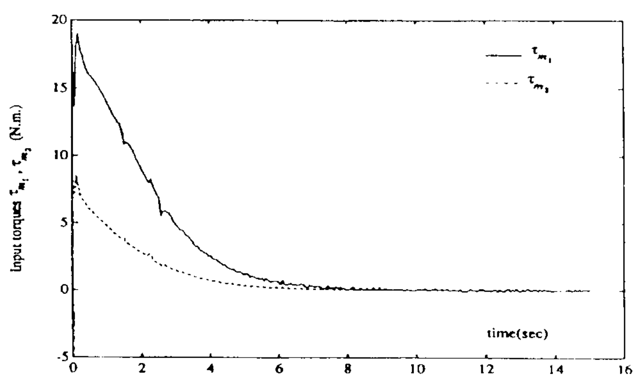

Fig. 14. Control effort for the closed loop system and nominal model; Al-Ashoor et al algorithm

on the rigid model, and a correction term designed based on the reduced flexible model. The robust stability of the PID controller is analyzed by lyapunov theory, and has been proven that the system is UUB stable. Then, the stability of the complete closed–loop system is analyzed and it is shown that the proposed controller is capable of robustly stabilizing the uncertain flexible joint manipulator. Finally, the effectiveness of the proposed control law is verified through simulations. Single, and two link flexible joint manipulators are examined in this study. The simulation results are compared to that given in the literature, and the effectiveness of preserving the robust stability, and performance of the system is verified and compared relative to them.

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