Design and Simulation of Robust Composite Controllers for Flexible Joint Robots

H.D. Taghirad and M.A. Khosravi
Advanced Robotics and Automated Systems (ARAS), Department of Electrical Engineering, K.N. Toosi U. of Technology, P.O. Box 16315-1355, Tehran, Iran. E-mail: taghirad@saba.kntu.ac.ir

Abstract

In this paper, the control of flexible joint manipulators is studied in detail. A composite control algorithm is proposed for flexible joint robots, which consists of two main parts. Fast control, \( u_f \), which guarantees that the fast dynamics remains asymptotically stable, and the corresponding integral manifold remains invariant. Slow control \( u_s \), itself consists of a robust PID designed based on the rigid model, and a corrective term designed based on the reduced flexible model. The stability of the overall closed-loop system is proved to be UUB stable, by Lyapunov stability analysis. Finally, the effectiveness of the proposed control law is verified through simulations. It is shown that the proposed control law ensures the robust stability and performance, despite the modeling uncertainties.

I. Introduction

After the inception of harmonic drive, multiple-axis flexible robot manipulators are widely used in industrial and space applications. In early eighties researchers showed that the use of control algorithms developed based on rigid robot dynamics on real non-rigid robots is very limited and may even cause instability [15]. To avoid this problem, many researchers have proposed control algorithms based on slow and fast dynamics of the system. Among them, many researchers in adaptive methods have proposed control algorithms based on the simple form of PID, and analyze the robust stability of the uncertain closed-loop system in the presence of structured and unstructured uncertainties. In this study, we introduce a new method based on the simple form of PID, and analyze the robust stability of the uncertain closed-loop system in the presence of structured and unstructured uncertainties. In this analysis, we introduce an integral manifold plus a composite control law in order to restrain the integral manifold invariant and to satisfy asymptotic stability requirement. The control effort consists of three elements, the first element is designed for the fast subsystem, the second term is a robust PID control designed for the rigid subsystem and the third term is a corrective term designed based on the first order approximation of the reduced flexible system. Based on the Lyapunov stability theory the complete closed-loop system is proven to be UUB stable. In order to verify the effectiveness of the proposed design method, and to compare its results to that presented in the literature, simulation of single and two link flexible joint manipulators are examined. It is shown in this study that the proposed control law ensures the robust stability and performance, despite the modeling uncertainties.

II. Flexible Joint Robot Modeling

Spong [13], has derived a nonlinear dynamical model for FJR using singular perturbation, in which the slow states are the position and velocities of the joints and the fast states are the forces and their derivatives. In order to model an N-axis robot manipulator with \( n \) revolute joints assume that: \( q_i : i = 1, 2, \ldots, n \) denote the position of \( i \)th link and \( \dot{q}_i : i = n + 1, n + 2, \ldots, 2n \) denote the position of the \( i \)th actuator scaled by the actuator gear ratio. If the joint is rigid \( q_i = \dot{q}_{n+i} \). For flexible joint, if the flexibility is modeled with a linear torsional spring with constant \( k_i \), the elastic force \( z_i \) is derived from:

\[
z_i = k_i (\dot{q}_i - \dot{q}_{n+i})
\]

The spring constants \( k_i \)'s are relatively large and rigidity is modeled by the limit \( k_i \rightarrow \infty \). Let \( u_i \) denotes the generalized force applied by the \( i \)th actuator and use the notation:

\[
q = (q_1, \ldots, q_n, \dot{q}_n, \dot{q}_{n+1}, \ldots, \dot{q}_{2n})^T = (q_1^T, \dot{q}_n^T)^T
\]

The equation of motion of the system can be written in the following form using Euler-Lagrange formulation:

\[
\begin{bmatrix}
M(q) \ddot{q}_i + N(q, \dot{q}_i) \\
J_i \ddot{q}_i = K(q_i - q_{i-1}) - D \dot{q}_i + T_F + u
\end{bmatrix}
\]

in which,

\[
N(q, \dot{q}_i) = \lambda \dot{q}_i + G(q_i) + F_c \dot{q}_i + F_e(q_i) + T_2
\]

and \( K \) is the joint stiffness matrix, \( M(q) \) is the mass matrix, \( \lambda \dot{q}_i \) is the matrix of Coriolis and centrifugal
terms, $G(q_1)$ is the vector of gravity terms, $F_d$ is the viscous friction matrix, $T_d$ is the vector of the joint bounded unmodeled dynamics, $J_m$ is the actuator moments of inertia matrix, $D$ is the actuator viscous friction matrix, and $T_p$ is the actuator bounded unmodeled dynamics. For all revolute manipulators, it is shown in [2, 10], that

$$m_1 I \leq M(q_1) \leq m_2 I ; \quad \|V_m(q, q)\| \leq C\|q_1\| \tag{5}$$

$$\|G(q_1)\| \leq C_1 ; \quad \|F_d q_1 + F_c(q_1)\| = C_{20} + C_{11} \|q_1\| \leq C_2 I \leq J_d I ; \quad d_1 I \leq D \leq d_2 I \tag{6}$$

Moreover, if the perturbations are bounded:

$$\|T_d\| \leq C_3 ; \quad \|T_f\| \leq C_4 \tag{7}$$

in which $C_{30}, C_{31}, C_4, d_3, d_4, q, q_0, q_1, C_{32}, m_3, m_5$ are positive real constants. If the joints are all rigid:

$$M(q)\ddot{q} + N(q, q) = u_0 \tag{9}$$

in which $q = q_1$ and $M$ is a positive definite matrix. This model is the model of FJR where $k \to 0$ verifying that the FJR model is a singularly perturbed model of rigid system. Assume that spring constants are equal the elastic forces of the springs can be calculated by:

$$e = k(q_1 - q_1), \quad K = kI \tag{10}$$

in order to use a small quantity for singular perturbation define $e = \frac{1}{k}$ by which for rigid system ($k \to \infty$) in this form we have $e \to 0$. Multiplying $M_1^{-1}$ to the both side of 2 and taking $z = k(q_1 - q_3)$, $q = q_1$, and using $q_3 = q_1 - e$:

$$\begin{align*}
\dot{q} &= a_1(q, q) + A_1(q)z \\
\dot{z} &= a_2(q, q) + A_2(q)z + B_2u
\end{align*} \tag{11}$$

in which,

$$A_1 = -M^{-1}(q); \quad a_1 = -M^{-1}(q)N(q, q) \tag{12}$$

$$A_2 = -eJ\dot{q} + J\ddot{q} - J^{-1}T_f - M^{-1}(q)N(q, q) \tag{13}$$

The reduced flexible model can be derived by replacing $z, \dot{z}$ with $H, \dot{H}$ in Equation 11.

$$\ddot{q} = a_1(q, q) + A_1(q)z \tag{14}$$

Equation 11 represents FJR as a nonlinear and coupled system. This representation includes both rigid and flexible subsystems in form of a singular perturbation model.

### III. Reduced Flexible Model

The singular perturbation model of the FJR is given in Equation 11. This model represents the flexibility in the joints, however, the reduced order model is the model of rigid system, which can be easily derived from Equation 11 by setting $e = 0$. With some matrix manipulation it can be shown that:

$$(M + J)\ddot{q} + N - T_f + D\dot{q} = u_0 \tag{15}$$

Rewrite this equation in this form:

$$M(q)\ddot{q} + N(q, q) = u_0 \tag{16}$$

in which

$$M(q) = M(q) + J \tag{17}$$

This representation introduces a 2n dimension manifold, $M_e$, which is called the rigid manifold. If $e \neq 0$ the produced manifold $M_e$, which is a function of $e$ represents the flexible system. To define flexible manifold $M_e$ assume:

$$z = H(q, q, u, e) \tag{18}$$

$$\dot{z} = H(q, q, u, e) \tag{19}$$

$M_e$ is an integral manifold for the flexible system if for each initial condition

$$\begin{align*}
z(t) &= \Delta \\
\dot{z}(t) &= \Delta' \quad \text{and} \quad q(t) = \zeta \\
\dot{q}(t) &= \zeta'
\end{align*} \tag{20}$$

in $M_e$ all trajectories of $q(t)$ and $z(t)$ for $t > t_0$ remain in the manifold $M_e$. In other words $\forall t > t_0$.

$$z(t) = H(q(t), q(t), u(t), e) \tag{21}$$

Now, the reduced flexible model can be derived by replacing $z, \dot{z}$ with $H, \dot{H}$ in Equation 11.

$$\ddot{q} = a_1(q, q) + A_1(q)H(q, q, u, e) \tag{22}$$

The order of this equation is equal to the rigid system, however, this model includes the effects of flexibility in form of an invariant integral manifold embedded in itself. Hence, this reduced order model is not an approximation of the FJR model, but it represents its projection on the integral manifold.

### IV. Composite Control

In order to have a valid reduced flexible model for the system, it is essential that the $M_e$ be an invariant manifold, or the fast dynamics be asymptotically stable. This can be satisfied using a composite control scheme [5]. In this framework the control effort $u$ consists of two main parts, $u_s$ the control effort for slow subsystem, and $u_f$ the control effort for fast subsystem, as:

$$u = u_s(q, q, e) + u_f(q, \dot{q}) \tag{23}$$

in which $u_f(q, \dot{q})$ is designed such that the fast dynamics becomes asymptotically stable. $u_s$ denotes the deviations of fast state variables from the integral manifold.

$$\eta = z - H(q, q, u, e) \tag{24}$$

$$\dot{\eta} = \dot{z} - \dot{H}(q, q, u, e) \tag{25}$$

The slow component of the control effort, $u_s(q, \dot{q}, e)$, is also designed based on the reduced flexible model. We describe the design technique for $u_f$ and $u_s$ in the next subsections, respectively.

#### A. Fast Subsystem Dynamics and Control

Recall Equation 24; hence,

$$\dot{\eta} = [a_2(q, \dot{q}, e\dot{z}) - a_2(q, \dot{q}, e\dot{H})] + A_2(q)\eta + B_2u_f \tag{26}$$

Substitute the value of $a_2$ and use fast time scale $\tau = \sqrt{\frac{m}{k}}$ with some manipulations we reach to [5]:

$$\dot{\eta} = A_2(\eta) + B_2u_f \tag{27}$$

and in state space form:

$$\begin{bmatrix} \dot{\eta} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A_2(q) \\ B_2 \end{bmatrix} u_f \tag{28}$$
The flexible modes are not stable since the eigenvalues are on the imaginary axis. Hence, \( u_f \) must be designed such that the eigenvalues are shifted to the left half plane in order to guarantee stability.

**Theorem 1:** The diagonal and positive definite matrices \( K_f \) and \( K_r \) exist such that the closed loop system including the subsystem 27 with the control effort \( u_f = K_f \eta + K_r \bar{\eta} \) becomes globally asymptotically stable. (Proof in [17])

**B. Control of Reduced Flexible Model**

The reduced flexible model represents the effect of flexibility in the form of the flexible integral manifold. In this section a robust control algorithm is proposed for the system based on this model. In order to accurately derive the control law \( u_f(q, \dot{q}, e) \) for the system, manipulation of partial differential equation is necessary. To avoid complex manipulations, we propose deriving the robust control law \( u_f(q, \dot{q}, e) \) from the series expansion of the integral manifold to the same order of \( \varepsilon \).

\[
H(q, \dot{q}, u_f, e) = H_0(q, \dot{q}, u_f) + \epsilon H_1(q, \dot{q}, u_f) + \cdots
\]

and implement the controller \( u_f(q, \dot{q}, e) \) in the same form as:

\[
u_f(q, \dot{q}, e) = \nu_0(q, \dot{q}) + \epsilon \nu_1(q, \dot{q}) + \cdots
\]

in which the functions \( H_i(q, \dot{q}, u_f) \), \( u_i(q, \dot{q}) \), \( i = 0, 1, \ldots \) are calculated iteratively without need to solve the partial differential equations. It is important to note that as \( \varepsilon \to 0 \), \( u_f \) tends to rigid control, and \( H \) tends to rigid integral manifold. \( \nu_0 \) is designed using a robust design technique based on the rigid reduced order model \( (\varepsilon = 0) \), and \( H_0 \) is calculated from:

\[
H_0 = -A_f^{-1}(a_{20} + B_2u_0)
\]

in which:

\[
a_{20} = a_2(q, \dot{q}, 0) = J^{-1}D\dot{q} - J^{-1}T_F(q, \dot{q}) - M^{-1}(q)N(q, \dot{q})
\]

Let:

\[
a_2(q, \dot{q}, \epsilon H) = a_{20} + \epsilon \Delta a_2 + O(\epsilon^2)
\]

in which \( a_{20} \) is given in Equation 32, and compare to Equation 13 we reach to:

\[
\begin{align*}
\Delta a_2 &= -J^{-1}DH \\
\Delta a_{20} &= -J^{-1}D\dot{H}_0
\end{align*}
\]

Hence,

\[
e\dot{H}_0 = a_{20} + A_2H_0 + B_2u_0 + e(\Delta a_{20} + A_2H_1 + B_2u_1) + O(\epsilon^2)
\]

and,

\[
\dot{H}_0 = \Delta a_{20} + A_2H_1 + B_2u_1
\]

Therefore,

\[
H_1 = A_f^{-1}(\dot{H}_0 - \Delta a_{20} - B_2u_1)
\]

To calculate \( u_1 \) refer to reduced flexible model 22 and approximate it to the first power of \( \epsilon \):

\[
\dot{q} = a_1(q, \dot{q}) + A_1(q)\dot{H}_0 + eA_1(q)A_f^{-1}(\dot{H}_0 - \Delta a_{20} - B_2u_1)
\]

By factoring the equal powers of \( \epsilon \) we reach to:

\[
u_1 = B_f^{-1}(\dot{H}_0 - \Delta a_{20})
\]

The only condition on robust control design is that \( u_0 \) must be at least twice differentiable. Finally, the control law for the subsystem has the form:

\[
u_s = u_0 + \epsilon u_1
\]

In which \( u_s \) is called the corrective term which is derived through this subsection and \( u_0 \) is the robust control based on the rigid model elaborated in the next section.

**C. Robust PID Control for Rigid model**

In this section we first propose a robust PID controller based on the rigid model of the system and then prove its robust stability with respect to the model uncertainties. Recall the rigid model of the system from Equation 15, choose a PID controller for \( u_0 \):

\[
u_0 = K_v \dot{e} + K_p e + K_r \int_0^t e(s)ds = Kz
\]

in which

\[
\begin{bmatrix}
e = q_e - q \\
K = [K_f \quad K_p \quad K_v] \\
\gamma = [\int_0^t e(s)ds \\ e^T \dot{\varepsilon}]^T
\end{bmatrix}
\]

Similar to [2, 11] and [12], assume:

\[
m_eI \leq M_e(q) \leq m_eI
\]

and put some limits on:

\[
||N_t|| \leq \beta_0 + \beta_1||L|| + \beta_2||L||^2 ; ||V_w|| \leq \beta_3 + \beta_4||L||
\]

in which ||.|| is the Euclidean norm and \( L = [e^T \quad \dot{\varepsilon}^T] \). Implement the control law \( u_0 \) in 15 to get:

\[
\dot{z} = Az + B\Delta A
\]

where

\[
A = \begin{bmatrix}
0 & I_n & 0 & 0 \\
0 & 0 & 0 & I_n \\
-M_t^{-1}K_i & -M_t^{-1}K_p - M_t^{-1}K_v \\
0 & 0 & 0 & 0
\end{bmatrix} \\
B = \begin{bmatrix}
0 \\
0 \\
M_t^{-1}
\end{bmatrix}
\]

\[
\Delta A = N_t + M_t\delta e
\]

To analyze the system robust stability consider the following Lyapunov function:

\[
V(z) = z^TPz = \frac{1}{2}[a_2 \int_0^t e(s)ds + a_1 e + e^T M_t e]
\]

\[
[a_2 \int_0^t e(s)ds + a_1 e + e^T] + w^TP_1w
\]

in which

\[
w = \int_0^t e(s)ds \\
P_1 = \frac{1}{2} \begin{bmatrix}
a_2 K_p + a_1 K_i + a_2 M_t & a_2 K_v + K_i \\
a_2 K_v + K_p + a_2 M_t & a_1 K_v + a_1 K_i + a_2 M_t
\end{bmatrix}
\]

Hence,

\[
F = \frac{1}{2} \begin{bmatrix}
a_2 K_p + a_1 K_i + a_2 M_t & a_2 K_v + K_i + a_2 M_t & a_2 M_t \\
a_2 K_v + K_p + a_2 M_t & a_1 K_v + a_1 K_i + a_2 M_t & a_1 M_t \\
a_2 M_t & a_1 M_t & M_t
\end{bmatrix}
\]

3110
Since $M_k$ is a positive definite matrix, $P$ is positive definite, if and only if, $P_t$ is positive definite.

**Lemma 1:** Assume the following inequalities hold:

\[ a_1 > 0, \quad a_2 > 0, \quad a_1 + a_2 < 1 \]
\[ s_1 = a_2(k_p - k) - (1 - a_1)k_t - a_2(1 + a_1 - a_2)m_t > 0 \]
\[ s_2 = k_p + (1 - a_2)k - k_t - a_1(1 + a_2 - a_1)m_t > 0 \]

Then $P$ is positive definite and satisfies the following inequality (Rayleigh-Ritz) [9]:

\[ \lambda(P)\|x\|^2 \leq V(x) \leq \lambda(P)\|x\|^2 \]

in which,

\[ \lambda(P) = \min \left\{ \frac{1 - a_1 - a_2}{2}, \frac{s_1}{s_2} \right\} \]
\[ \lambda(P) = \max \left\{ \frac{1 + a_1 + a_2}{2}, \frac{s_3}{s_4} \right\} \]

and

\[ s_3 = a_2(k_p + k_v) + (1 + a_1)k_t + (1 + a_1 + a_2)a_2m_t \]
\[ s_4 = a_1m_t(1 + a_1 + a_2) + (a_1 + a_2)k + k_t \]

Proof is based on Gershgorin theorem and is similar to that in [11]. With some manipulations we can show [5]:

\[ \gamma = \min \{a_2k_t, a_1k_p - a_2k_v - k_t, k_v\} \]

Now considering Equations 40, 42 and $\|L\| \leq \|x\|$ then,

\[ \xi_0 = \gamma - \lambda_1\beta_0 - \lambda_2\gamma\lambda_3m_t \]
\[ \xi_1 = \gamma - \lambda_1\beta_0 - \lambda_2\gamma\lambda_3\beta_1 \]
\[ \xi_2 = \lambda_1\beta_4 + \alpha_2^\top \lambda_1\beta_1 \]

in which

\[ \lambda_1 = \lambda_{\max}(R_1) \]
\[ \lambda_2 = \lambda_{\max}(R_2) \]
\[ \lambda_3 = \sup \|\dot{\bar{q}}\| \]

and $\lambda_{\max}, \lambda_{\min}$ are the least and largest eigenvalues, respectively, and

\[ R_1 = \begin{bmatrix} a_2^\top I & a_1a_2^\top I & a_1^\top I \\ a_0a_2^\top I & a_0^\top I & a_0^\top I \\ a_2^\top I & a_2^\top I & a_2^\top I \end{bmatrix} \]
\[ R_2 = \frac{1}{2} \begin{bmatrix} a_2^\top I & a_2^\top I & a_2^\top I \\ a_0a_2^\top I & a_0^\top I & a_0^\top I \\ a_2^\top I & a_2^\top I & a_2^\top I \end{bmatrix} \]

According to the result obtained so far, we can prove the stability of the error system based on the following theorem.

**Theorem 2:** The error system 41 is stable of the form of UUB, if $\xi_1$ is chosen large enough.

The conditions are:

\[ \xi_1 > 2\sqrt{\xi_0\xi_2} \]
\[ \xi_1^2 + \xi_1\sqrt{\xi_1^2 - 4\xi_0\xi_2} > 2\xi_0\xi_2(1 + \sqrt{\frac{\lambda(P)}{\lambda(P)}}) \]
\[ \xi_1 + \sqrt{\xi_1^2 - 4\xi_0\xi_2} > 2\xi_0\|\dot{\bar{q}}\| \sqrt{\frac{\lambda(P)}{\lambda(P)}} \]

These conditions can be simply met by making $\xi_1$ large enough by choosing large enough control gains $K_p, K_v,$ and $K_t.$ (Proof in [17])

Fig. 2. Poor tracking performance of the closed-loop system for perturbed model, Spong algorithm.

**V. Stability Analysis of the Complete Closed-loop System**

The stability of the fast, and slow subsystems are separately analyzed in previous sections. However, the stability of the complete closed-loop system may not be guaranteed through these separate analysis [8]. In this section the stability of the complete system is analyzed. Recall the dynamic equations of the FJR Equation 11. The integral manifold and the control effort are chosen as:

\[ \eta = z - H \]
\[ H = H_0 + \epsilon_1H_1 \]
\[ u = u_0 + u_f = u_0 + \epsilon_2u_1 + u_f \]

Combine these equations to Equation 11, 35, 31 and 38, and consider, $x = \left[ \int_0^t e(s) \, ds \quad e^2 \quad \dot{e}^2 \right]^T.$ \[ y = \left[ \eta^T \quad \eta^T \right]^T \]

\[ \dot{x} = Ax + B\Delta A + C[I \quad 0]y \]
\[ e\gamma = \tilde{A}y \]

in which

\[ A = \begin{bmatrix} \emptyset & I & 0 \\ 0 & \emptyset & I \\ \emptyset & 0 & \emptyset \end{bmatrix} \]
\[ B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \]
\[ \Delta A = N_t + M\hat{q}_d \]
\[ C = \begin{bmatrix} \emptyset \\ \emptyset \\ -A_1 \end{bmatrix} \]

\[ \dot{\tilde{A}} = \begin{bmatrix} \emptyset \\ \emptyset \\ A_2 + B_2K_{sf} \end{bmatrix} \]

\[ \xi = \xi - 2J^{-1}D + B_2K_{sf} \]

**Theorem 3:** There exist diagonal and positive definite matrices $K_{sf}$ and $K_{vf}$ such that the closed loop system 46 becomes globally asymptotically stable. (Proof in [17])

**Theorem 4:** The closed-loop system of Equations 45 and 46 is UUB stable if $K_{sf}, K_{vf},$ and $\xi_1$ are chosen large enough. (Proof in [17])

The detail conditions on the PID controller parameter bounds to preserve the closed-loop stability, are given in [17]. However, the stability conditions met if the controller gains are selected large enough.

**VI. Simulations**

In order to verify the effectiveness of the algorithm a simulation study has been forwarded next. In the following simulation study, the results of the closed loop performance of a single, [14], and a two link flexible joint manipulator, [1], examined in the literature is compared to that of the proposed control algorithm.
A. Single Link Flexible Joint Manipulator

Consider the single link flexible joint manipulator introduced in [14]. The dynamic equation of motion of this system is as following:

$$
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{M g L}{I} \sin(x_1) - \frac{K}{I} (x_1 - x_3) \\
\dot{x}_3 &= \frac{K}{J} (x_1 - x_3) + \frac{1}{J} u \\
\dot{x}_4 &= \frac{K}{J} (x_1 - x_3) + \frac{1}{J} u
\end{align*}
$$

in which $x_1 = q$ and $x_2 = \dot{q}$. By choosing $q_1 = q$ and $z = K(q_1 - q_2)$ as the elastic force, the model of the system can be rewritten in a singular perturbation form:

$$
\begin{align*}
\dot{\bar{q}} &= -M g L \sin(q) - \frac{1}{J} \bar{z} \\
\epsilon \dot{\bar{z}} &= -\frac{M g L}{I} \sin(q) - \left( \frac{1}{J} + \frac{1}{J} \right) \bar{z} - \frac{1}{J} u
\end{align*}
$$

in which $\epsilon = \frac{1}{J}$.

Spong has proposed a composite control law for this system, in which there exists two control components corresponding to the fast and slow dynamics. As it is illustrated in [14], the closed loop system became unstable, provided that only the corresponding rigid control effort $u_0$ is applied on the system. Moreover, the system becomes stable and the desired trajectory $q_d = \sin(\theta t)$ is well tracked, implementing the proposed composite control on the nominal model of the system. However, this algorithm is not robust to the model parameter variations. As illustrated in Figure 2, the tracking performance is getting quite poor for the maximum perturbation values for the parameters $I, J, M$, and $L$. For the sake of comparison, the proposed robust PID controller may be now applied on the same system. The proposed control law is composed of three terms, in which the rigid control law is a PID controller whose coefficients satisfies the robust stability conditions elaborated in Theorem (4) as following:

$$
u_0 = 200 \bar{e} + 500 \bar{e} + 100 \int_0^t e(s)ds.
$$

The integral manifold would be:

$$
H_0 = -4.9 \sin(q) - \frac{1}{2} \nu_0
$$

and the corrective term corresponds to

$$
u_1 = H_0.
$$

B. Multiple Link Flexible Joint Manipulator

Consider the two link Flexible Joint manipulator illustrated in Figure 1. In this manipulator Joint flexibility is modeled with a linear torsional spring with stiffness $K$. The equation of motion of this system and its parameters is given in [1]. Our proposed algorithm is applied to the system for comparison of the results. Hence, the reduced order first order model is evaluated as following:

$$
\begin{bmatrix}
2.25 & 0.5 \cos(\epsilon H_1^e) \\
0.5 \cos(\epsilon H_1^e) & 1.25
\end{bmatrix}
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2
\end{bmatrix}
+ \begin{bmatrix}
H_1^e + \epsilon H_1^e \\
H_2^e + \epsilon H_2^e
\end{bmatrix}
= \begin{bmatrix}
14.7 \cos(\theta) + 0.5 \dot{\theta}_1 \sin(\epsilon H_1^e) \\
4.9 \cos(\theta) - 0.5 \dot{\theta}_1 \sin(\epsilon H_1^e)
\end{bmatrix}
$$

In order to evaluate the fast dynamics caused by the joint flexibility, the normalized time variable $\tau = \frac{1}{J} t$ is used. Hence,

$$
H_1^e = \dot{\theta}_1 - u_1^0 \quad H_2^e = \dot{\theta}_2 - u_2^0
$$

With expanding Equation 49 to the first order of $\epsilon$ we have:

$$
H_1^e = -5.0 \bar{\theta}_1 \quad H_2^e = -5.0 \bar{\theta}_2
$$

And from Equation 50 we get:

$$
u_1 = -5.0 \dot{\bar{\theta}}_1 \quad \nu_2 = -5.0 \dot{\bar{\theta}}_2
$$

Finally, the slow part of the control law will be calculated from:

$$
u_{1s} = u_1 + cu_1; \quad u_{2s} = u_2 + cu_2
$$

The $u_1, u_2$ are the rigid part of the control law and as elaborated before is robustly designed as a PID controller.
In here we design the PID gains as following which satisfies the robust conditions:

$$u_1^* = 500e + 50e + 50 \int_0^t e(s)ds$$
$$u_2^* = 200e + 50e + 50 \int_0^t e(s)ds$$

(54)

The fast control law is also designed as a PD controller as:

$$u_1 = \theta_1 + \eta_1 ; u_2 = \theta_2 + \eta_2$$

(55)

Finally the control law is composed from the fast and slow parts:

$$u_1 = u_1^* + u_1; u_2 = u_2^* + u_2$$

(56)

To have simulation results compared to [1], the reference signal is considered as:

$$\theta_i = 1.57 + 7.8539e^{-t} - 9.428e^{-t/2} i = 1, 2$$

(57)

in which the joint angles reach to a final value of $\theta_i = \frac{\pi}{2}$ from zero initial state. Figure 4 illustrates the response of the perturbed system to our proposed composite control law. The system becomes stable, and the tracking performance is quite desirable despite the 50% variation in model parameters. The control is limited to a maximum allowable bound by adding a saturation block in the simulation. Al-Ashoor et al have used a robust-adaptive control law in addition to the composite law we introduced in this paper. Figure 5 illustrates the results obtained for the reference signal introduced in Equation 57 in [1]. This figure illustrates the tracking performance despite the bounded control effort illustrated in Figure 6. Comparing it to our result (Figure 4), similar performances are obtained. The only limitation exists in our proposed law compared to that in [1], is the amplitude of the control law in the initial time of the simulation. The adaptive law have smaller control effort in the beginning of the simulation, due to the adaptive nature of the algorithm, and using the information of the identified model of the system in the control law. This issue is under current investigation, and promising results are obtained by a $H_{\infty}$-based composite controller, in which the control effort can be limited to desirable bounds, [16].

VII. Conclusions

In this paper the control of flexible joint manipulators is examined in detail. In order to achieve the required performance a composite control algorithm is proposed, consisting of corresponding control law for fast and slow subsystems. A simple PD control is proposed for the fast subsystem, and it is proven that the fast subsystem becomes asymptotically stable. The slow subsystem itself is controlled through a robust PID controller designed based on the rigid model, and a correction term designed based on the reduced flexible model. The stability of the complete closed-loop system is analyzed and it is shown that the proposed controller is capable of robustly stabilizing the uncertain joint manipulator. Finally, the effectiveness of the proposed control law is verified through simulations, are compared to that given in the literature, and the effectiveness of preserving the robust stability, and performance of the system is verified and compared relative to them.

References