Robust Stability Analysis of FJR Composite Controller with a Supervisory Loop

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***Abstract* - In this paper a controller design method for flexible joint robots (FJR), considering actuator saturation is proposed and its robust stability is thoroughly analyzed. This method consists of a composite control structure, with a PD controller on the fast dynamics and a PID controller on slow dynamics. Moreover, the need of powerful actuator is removed by decreasing the bandwidth of the fast controller during critical occasions, with the use of a supervisory loop. Fuzzy logic is used in the supervisory law, in order to adjust the proper gain in the forward path. It is then shown that UUB stability of the overall system is guaranteed in presence of uncertainties, provided that the PD and the PID gains are tuned to satisfy certain conditions.**

Index Terms - Flexible joint robot, Actuator saturation, Supervisory control, Stability analysis, Fuzzy logic..

I. INTRODUCTION

The desire for higher performance from the structure and mechanical specifications of robot manipulators has spurred designers to come up with *flexible joint robots* (FJR). This necessity has emerged new control strategies required, since the traditional controllers implemented on FJRs have failed in performance [1], [2]. Since 1980's many attempts have been made to remedy this shortcoming and now, several methods has been proposed including various linear, nonlinear, robust, adaptive and intelligent controllers [3], [4].However, the practical limitations such as actuator saturation is rarely considered in the controller synthesis, as an important practical drawback to achieve good performance [5]. On the other hand actuator saturation has been considered by the control community from early achievements of control engineering. A common classical remedy for systems with bounded control is to reduce the bandwidth of the control system such that saturation seldom occurs. This is a trivial weak solution, since even for small reference commands and disturbances the possible performance of the system is significantly degraded. This idea of reduction in bandwidth by reduction in the closed loop gain, is easily implementable, therefore, this motivates some researchers to propose an "adaptive" reduction in bandwidth [6]. The "adaptation" process is done under supervision of a *supervisory loop*, and as proposed in [6] can be accomplished through complex computations. In order to come up with an online implementable controller for FJRs, a fuzzy logic supervisory control has been proposed by authors in [7]. In this topology, the fuzzy logic is set to be "out of the main loop", at a supervisory level, at the aim of preserving the essential properties of the main controller. This idea is first published by the authors in [8] and is modified to use with composite controller for FJRs in [7]. It is observed in various simulations that by including this supervisory loop to the controller structure, the steady state performance of the system is preserved, and moreover, the stability of the overall system is preserved. The stability analysis of the overall system, however, is essential for the closed loop structure for susceptible applications of the FJRs such as space robots, where the stability is a main concern. This issue is analyzed thoroughly in this paper.

In this paper a PD controller is used to stabilize the fast dynamics and a PID controller is proposed to robustly stabilize the slow dynamics for the FJR. Moreover, a supervisory control loop is added to the structure in order to decrease the bandwidth of the fast controller during critical occasions. The robust stability of the overall system in presence of the modeling uncertainties is then analyzed in detail, and it is shown that UUB stability of the overall system is guaranteed, only if the PD gains of the fast controller and the PID gains are tuned to satisfy certain conditions.

II. FJR MODELING

In order to model an FJR, the state vector includes the link positions, and the actuator positions in a vector as follows:

$$\vec{Q} = [\theta_1, \theta_2, ..., \theta_n, \theta_{n+1} \,|\, \theta_{n+2}, ..., \theta_{2n}]^T = [\vec{q}_1^{\,T}, \vec{q}_2^{\,T}]^T \tag{1}$$

Where $\theta_i : i = 1,2,...,n$ represents the position of the i'th link and the position of the i'th actuator is represented by $\theta_{i+n} : i=1,2,...,n$. Using this notation and considering some simplifying assumptions [9], the governing equation of motion of the system is as follows:

$$\begin{cases} \mathbf{M}(\vec{q}_1)\ddot{\vec{q}}_1 + \vec{N}(\vec{q}_1, \dot{\vec{q}}_1) = -\mathbf{K}(\vec{q}_1 - \vec{q}_2) \\ \mathbf{J} \quad \ddot{\vec{q}}_2 - \mathbf{K}(\vec{q}_1 - \vec{q}_2) = \vec{u} \end{cases} \tag{2}$$

Where **M** is the matrix of the link inertias and **J** is that of the motors, $\vec{u}$ is the vector of input torques and $\vec{N}$ is the vector of all gravitational, centrifugal and Coriolis torques as follows:

$$\vec{N}(\vec{q}_1, \dot{\vec{q}}_1) = \mathbf{V_m}(\vec{q}_1, \dot{\vec{q}}_1)\,\dot{\vec{q}}_1 + \vec{G}(\vec{q}_1) + \mathbf{F_d}\dot{\vec{q}}_1 + \vec{F}_S(\dot{\vec{q}}_1) + \vec{T}_d \tag{3}$$

In which matrix $\mathbf{V_m}(\vec{q}_1, \dot{\vec{q}}_1)$ consists of the Coriolis and centrifugal terms, G(q) is the gravity terms, $\mathbf{F_d}$ is the diagonal

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viscous friction matrix, F_s includes the static friction terms, and finally, T_d is the vector of disturbance and unmodeled but bounded dynamics. Including this last term in the model, enables us to encapsulate the modeling uncertainties into the picture. As it is demonstrated in [10], the following quantities are all bounded:

$$\underline{m}\ \ \mathbf{I} \le \mathbf{M}(\bar{q}_1) \le \overline{m}\ \ \mathbf{I} \tag{4}$$

$$\left\|\bar{N}(\bar{q}_1,\dot{\bar{q}}_1)\right\| \le \beta_0 + \beta_1\left\|\dot{\bar{q}}_1\right\| + \beta_2\left\|\dot{\bar{q}}_1\right\|^2 \tag{5}$$

$$\left\|\mathbf{V_m}(\bar{q}_1,\dot{\bar{q}}_1)\right\| \le \beta_3 + \beta_4\left\|\dot{\bar{q}}_1\right\| \tag{6}$$

Where $\underline{m},\overline{m}$ and β_i's are real positive constants and these uncertainty bounds will be used in robust stability analysis. It is assumed that all flexible elements are modeled by linear springs and without loss of generality [7], all springs assumed to have the same spring constant k. the matrix $\mathbf{K}$ is defined as $\mathbf{K} = k\mathbf{I}$ .The inertia matrices are non-singular so the model can be changed to the following singular perturbation standard form:

$$\begin{cases} \ddot{\bar{q}} = -\mathbf{M}^{-1}(\bar{q})\bar{z} - \mathbf{M}^{-1}(\bar{q})\bar{N}(\bar{q},\dot{\bar{q}}) \\ \varepsilon\,\ddot{\bar{z}} = -(\mathbf{M}^{-1}(\bar{q}) + \mathbf{J}^{-1})\bar{z} - \mathbf{M}^{-1}(\bar{q})\bar{N}(\bar{q},\dot{\bar{q}}) - \mathbf{J}^{-1}\bar{u} \end{cases} \tag{7}$$

in which $\bar{q} = \bar{q}_1$, $\bar{z} = \mathbf{K}(\bar{q}_1 - \bar{q}_2)$ and $\varepsilon = 1/k$. Now if we choose $\varepsilon = 0$ then the slow behavior of $\bar{z}$ could be derived as:

$$\begin{aligned} \bar{z}_S &= -(\mathbf{M}^{-1}(\bar{q}) + \mathbf{J}^{-1})^{-1}[\mathbf{M}^{-1}(\bar{q})\bar{N}(\bar{q},\dot{\bar{q}}) - \mathbf{J}^{-1}\bar{u}_S] \\ &= -(\mathbf{I} + \mathbf{M}(\bar{q})\mathbf{J}^{-1})^{-1}\bar{N}(\bar{q},\dot{\bar{q}}) - (\mathbf{J}\ \ \mathbf{M}(\bar{q})^{-1} + \mathbf{I})^{-1}\bar{u}_S \end{aligned} \tag{8}$$

By substitution of this into equation (7) after some matrix manipulations we reach to:

$$\ddot{\bar{q}}_S = -\mathbf{M}^{-1}(\bar{q})\bar{z}_f - [\mathbf{J} + \mathbf{M}(\bar{q})]^{-1}\bar{N}(\bar{q},\dot{\bar{q}}) + [\mathbf{J} + \mathbf{M}(\bar{q})]^{-1}\bar{u}_S \tag{9}$$

in which $\bar{z}_f$ represents the fast behavior of $\bar{z}$ which is defined as $\bar{z}_f = \bar{z} - \bar{z}_S$. Its dynamics could be found to be

$$\varepsilon\ \ \ddot{\bar{z}}_f = -(\mathbf{M}^{-1}(\bar{q}) + \mathbf{J}^{-1})\bar{z}_f - \mathbf{J}^{-1}\bar{u}_f \tag{10}$$

having $\bar{u}_S$ and $\bar{u}_f$ we can solve the last three equations to find $\bar{q}_S, \bar{z}_S$ and $\bar{z}_f$ .Tikhonov theorem [11], provides some stability conditions, under which, the overall behavior of the system can be determined from these variables as follows:

$$\bar{q}(t) = \bar{q}_S(t) + O(\varepsilon) \qquad t \in [0,T]$$

$$\bar{z}(t) = \bar{z}_S(t) + \bar{z}_f(t) \qquad t \in [0,T]$$

$$\exists\ \ t_1 \ni \bar{z}(t) = \bar{z}_S(t) + O(\varepsilon) \qquad t \in [t_1,T]$$

In which, $O(\varepsilon)$ determines the terms whose order is as the order of ε The stability conditions can be satisfied by proper selection of $\bar{u}_f$. Taking into account the dynamics of $\bar{z}_f$ (equation (7)), a proper second order dynamics can be imposed to it by the use of a simple PD controller:

$$\bar{u}_f = [\mathbf{K}_{\mathbf{Pf}}\bar{z}_f + \mathbf{K}_{\mathbf{Df}}\dot{\bar{z}}_f] \tag{11}$$

Different control strategies can be used for the slow subsystem. Using a robust PID controller for the $\bar{u}_S$ provides the following benefits: No need for rate measurements, no need for offline computations, and guaranteed robust stability under the conditions detailed in [12]. These characteristics makes this structure attractive for practical implementation, hence, we propose using such structure.

$$\bar{u}_S = \mathbf{K_P}\bar{e} + \mathbf{K_D}\dot{\bar{e}} + \mathbf{K_I}\int_0^t \bar{e}(\tau)d\tau \tag{12}$$

in which the error vector is defined as:

$$\bar{e} = \bar{q}_d - \bar{q} \tag{13}$$

The overall control system is shown in Fig. (1) by which the desirable performance can be achieved at the expense of high control effort, and may result in actuator saturation. This draw back is remedied by a supervisory loop added to the control structure which will be detailed in the proceeding section.

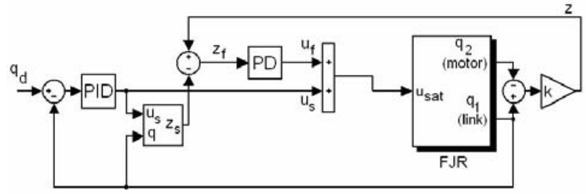

Fig. 1 The FJR control system

III. The Supervisory Loop

Let us first describe the idea of error governor as it is first proposed by the authors in [6]. Then the modifications needed to use this idea with the FJR model are elaborated. Without loss of generality one can assume that each element $u_i(t)$ of the control vector has a saturation limit of 1. In other words the saturation function can be defined as follows:

$$sat(u_i(t)) = \begin{cases} 1 & 1 \le u_i(t) \\ u_i(t) & -1 \le u_i(t) \le 1 \\ -1 & u_i(t) \le -1 \end{cases} \tag{14}$$

The proposed method is twofold, first the compensator is designed without considering any saturation limit, then a time varying scalar gain $0<\lambda(t)\le 1$ is added which modifies error and is adjusted via a supervisory loop in order to cope with saturation as depicted in Fig. 2.

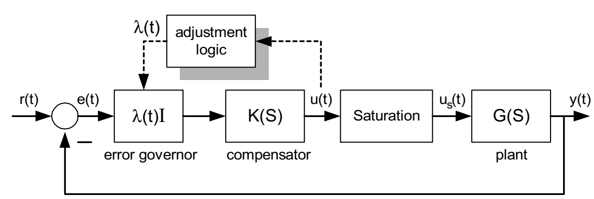

Fig. 2. The closed loop system with error governor

Intuitively one can state the logic of adjustment as follows:

- If the system is to experience saturation make λ smaller,
- Otherwise increase λ up to one.

This logic decreases the bandwidth when the system is to experience saturation and in normal conditions the effect of error governor is diminished by making $\lambda=1$. This configuration reduces the amplitude of the control effort as is done by saturation itself but there are some important differences. First, this is a dynamic compensator and not a hard nonlinearity as is the case with saturation. Second, this approach limits the control effort by affecting the controller states while saturation will limit the control effort independent

of the controller states. In other words, it acts in a closed loop fashion rather than an open loop structure of a saturation block. Hence, the dynamic behavior of it can be used to preserve stability.
It is difficult to implement this logic with a rigorous mathematical model. However, fuzzy logic can be easily employed in here as first proposed by authors in [8]. Details are as follows. In order to sense the value of closeness to saturation, the absolute value of the amplitude of the control effort $|u(t)|$ can be used as a good measure. To give a kind of prediction to the logic $\dot{u}(t)$ is also taken into account. The above logic thus can be interpreted with fuzzy notation as follows:

- If $|u(t)|$ is *NEAR* to one and $\dot{u}(t)$ is *POSITIVE* make λ *LESS* than one,
- When $|u(t)|$ is *OVER* one, make λ *SMALL* if $\dot{u}(t)$ is negative and *VERY SMALL* if $\dot{u}(t)$ is not negative,
- Otherwise make it *ONE* (see table 1).

Table 1 Fuzzy Rules

$\dot{u}$ \ $\lvert u\rvert$	**Small**	**Near**	**Over**
Neg	**One**	**One**	**S**
Zero	**One**	**One**	**VS**
Pos	**One**	**L**	**VS**

To implement this logic, corresponding fuzzy sets are defined. Since the logic is based on a model free routine, the proposed method can be implemented not only on FJRs but also on a variety of systems experiencing limitations in the actuators, and the effectiveness of this structure is verified in different applications [7], [8].

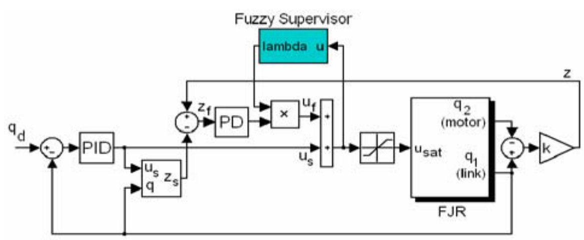

Figure 3: The complete control system for the FJR with fuzzy supervisor

In order to use this strategy for the FJR, some adjustments to the general structure has been made, which is briefed as follows. First, the supervisor is only applied for the fast subsystem, which mainly causes the instability when limited by saturation. Second, the saturation limit is not 1 in the FJR configuration, so the control effort u(t) must be attenuated by this factor before feeding to the supervisor. The modified supervisory loop for the FJR is shown in Fig. 3. A filter is used to estimate $\dot{u}(t)$ from $u(t)$ so that the only measurement required is $u(t)$. As mentioned in the previous section the composite controller composed of a fast PD controller and a slow PID controller has been shown to be robustly stable [12]. But the robust stability of the system after adding a fuzzy supervisor and the effect of the term λ(t) on it is studied in this paper, and in the following section.

IV. Robust Stability Analysis

Recall the system equations

$$\ddot{\bar q}_S = -\mathbf{M}^{-1}(\bar q)\bar z_f - \mathbf{M}_t^{-1}(\bar q)\ \bar N(\bar q,\dot{\bar q}) + \mathbf{M}_t^{-1}(\bar q)\ \bar u_S \qquad (15)$$

$$\varepsilon\ddot{\bar z}_f = -(\mathbf{M}^{-1}(\bar q)+\mathbf{J}^{-1})\bar z_f - \mathbf{J}^{-1}\bar u_f \qquad (16)$$

Where $\mathbf{M}_t(\bar q) = \mathbf{J}+\mathbf{M}(\bar q)$ and the control terms in presence of supervisor have the form of:

$$\bar u_S = \mathbf{K_P}\bar e + \mathbf{K_D}\dot{\bar e} + \mathbf{K_I}\int_0^t \bar e(\tau)d\tau \qquad (17)$$

$$\bar u_f = \lambda(t).[\mathbf{K_{Pf}}\bar z_f + \mathbf{K_{Df}}\dot{\bar z}_f] \qquad (18)$$

the error dynamics could be evaluated as:

$$\begin{aligned}\ddot{\bar e} &= \ddot{\bar q}_d + \mathbf{M}^{-1}(\bar q)\bar z_f + \mathbf{M_t}^{-1}(\bar q)\mathbf{N}(\bar q,\dot{\bar q}) \\ &\quad -\mathbf{M_t}^{-1}(\bar q)(\mathbf{K_P}\bar e + \mathbf{K_D}\dot{\bar e} + \mathbf{K_I}\int_0^t \bar e(\tau)d\tau)\end{aligned} \qquad (19)$$

and the fast dynamics is

$$\varepsilon\ddot{\bar z}_f = -(\mathbf{M}^{-1}(\bar q)+\mathbf{J}^{-1})\bar z_f - \mathbf{J}^{-1}\lambda(t).[\mathbf{K_{Pf}}\bar z_f + \mathbf{K_{Df}}\dot{\bar z}_f] \qquad (20)$$

We can rewrite this dynamics as

$$\ddot{\bar z}_f = -\mathbf{K_1}\dot{\bar z}_f - \mathbf{K_2}\bar z_f \qquad (21)$$

in which

$$\mathbf{K_1} = \frac{\lambda(t)}{\varepsilon}\mathbf{J}^{-1}\mathbf{K_{Df}} \qquad (22)$$

$$\mathbf{K_2} = (1/\varepsilon)[\mathbf{M}^{-1}(\bar q) + \mathbf{J}^{-1}(\mathbf{I}+\lambda(t)\mathbf{K_{Pf}})] \qquad (23)$$

These equations can be rearranged into state space format as

$$\dot{\bar x} = \mathbf{A}\ \ \bar x + \mathbf{B}\Delta\mathbf{A} + \mathbf{C}[\mathbf{I}\quad \mathbf{0}]\bar y \qquad (24)$$

$$\dot{\bar y} = \mathbf{A_f}\bar y \qquad (25)$$

in which

$$\bar x = \begin{bmatrix}\int_0^t \bar e(\tau)d\tau \\ \bar e \\ \dot{\bar e}\end{bmatrix},\quad \bar y = \begin{bmatrix}\bar z_f \\ \dot{\bar z}_f\end{bmatrix} \qquad (26)$$

and

$$\mathbf{A} = \begin{bmatrix}\mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ -\mathbf{M_t}^{-1}(\bar q)\mathbf{K_I} & -\mathbf{M_t}^{-1}(\bar q)\mathbf{K_P} & -\mathbf{M_t}^{-1}(\bar q)\mathbf{K_D}\end{bmatrix}, \mathbf{B} = \begin{bmatrix}\mathbf{0} \\ \mathbf{0} \\ \mathbf{M_t}^{-1}(\bar q)\end{bmatrix}, \qquad (27)$$

$$\Delta\mathbf{A} = (\mathbf{N}(\bar q,\dot{\bar q}) + \mathbf{M_t}(\bar q)\ddot{\bar q}_d),\quad \mathbf{C} = \begin{bmatrix}\mathbf{0} \\ \mathbf{0} \\ \mathbf{M}^{-1}(\bar q)\end{bmatrix}$$

$$\mathbf{A_f} = \begin{bmatrix}\mathbf{0} & \mathbf{I} \\ -\mathbf{K_2} & -\mathbf{K_1}\end{bmatrix} \qquad (28)$$

In the next subsection we study the effect of λ(t) on the stability of the fast subsystem.

A. *Stability of the fast subsystem*

To study the stability of the fast subsystem we consider the following Lyapunov function candidate:

$$V_f(\bar y) = \bar y^T\mathbf{S}\ \ \bar y \qquad (29)$$

in which $\mathbf{S}$ is defined as

$$\mathbf{S} = 1/2\begin{bmatrix}2\mathbf{I} & \mathbf{K_1}^{-1} \\ \mathbf{K_1}^{-1} & \mathbf{K_2}^{-1}\end{bmatrix} \qquad (30)$$

Lemma 1: *The matrix* **S** *is positive definite.*

The proof is based on Shur complement [13] and is the same as what can be found in [14] for the case $\lambda(t) = 1$. So the function V_f is positive. ■

Theorem 1: The fast subsystem of (25) with the matrix $\mathbf{A_f}$ introduced in (28) is stable provided that condition (33) is met. In other words, there are some bounds on the parameters $\mathbf{K_{PF}}$, $\mathbf{K_{DF}}$, $\lambda(t)$ used in control term (18) which if it is satisfied, the dynamics (16) in a closed loop configuration becomes stable.

Proof: To prove stability using Lyapunov direct method consider the time derivative of V_f along trajectory (25)

$$\begin{aligned}\dot V_f(\bar y) &= \dot{\bar y}^T \mathbf{S}\ \bar y + \bar y^T \mathbf{S}\ \dot{\bar y} + \bar y^T \dot{\mathbf{S}}\ \bar y \\ &= \bar y^T[\mathbf{A_f}^T\mathbf{S} + \mathbf{S}\ \mathbf{A_f}]\bar y + \bar y^T \dot{\mathbf{S}}\ \bar y\end{aligned} \tag{31}$$

considering the first term, note that

$$\mathbf{A_f}^T\mathbf{S} + \mathbf{S}\ \mathbf{A_f} = -\begin{bmatrix}\mathbf{K_1}^{-1}\mathbf{K_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{K_1}\mathbf{K_2}^{-1} - \mathbf{K_1}^{-1}\end{bmatrix} = -\mathbf{W} \tag{32}$$

Since matrices $\mathbf{K_1}$ and $\mathbf{K_2}$ are positive definite, in order to make matrix $\mathbf{W}$ positive definite the following matrix should be positive definite.

$$\mathbf{K_1}\mathbf{K_2}^{-1} - \mathbf{K_1}^{-1} > \mathbf{0} \tag{33}$$

After some matrix manipulations this can be transformed to the following condition on PD gain matrices

$$\frac{1}{\varepsilon}\left[\underline{\lambda}(\lambda(t)\mathbf{J}^{-1}\mathbf{K_{Df}})\right]^2 > \bar\lambda(\mathbf{M}^{-1}(\bar q) + \mathbf{J}^{-1}[\mathbf{I} + \lambda(t)\mathbf{K_{Pf}}]\ ) \tag{34}$$

Where $\underline{\lambda}$ and $\bar\lambda$ represent the smallest and largest eigenvalues, respectively. Note that, if we let $\lambda(t) = 1$ this can be met by increasing $\mathbf{K_{DF}}$ and decreasing $\mathbf{K_{PF}}$. In other words a lower bound on $\mathbf{K_{DF}}$ and simultaneously an upper bound on $\mathbf{K_{PF}}$ should be satisfied (as is the case when there is not a supervisory loop). For example if we assume the gain matrices to be diagonal

$$\mathbf{K_{Df}} = k_{Df}\mathbf{I}\ , \quad \mathbf{K_{Pf}} = k_{Pf}\mathbf{I} \tag{35}$$

then the following conditions must be met in absence of supervisor.

$$k_{Df} \ge \underline{k}, \ k_{Pf} \le \bar k \tag{36}$$

Where $\underline{k}$ and $\bar k$ are some positive constants. In presence of supervisory logic, as $\lambda(t)$ is always less than 1, the upper bound can be left unchanged. However, the lower bound can be adjusted assuming a lower bound on $\lambda(t)$

$$\lambda(t) > \Lambda_{\min} \tag{37}$$

then:

$$k_{Df} \ge \frac{\underline{k}}{\Lambda_{\min}} \tag{38}$$

Now consider the second term in equation (31)

$$\dot{\mathbf{S}} = -\tfrac{1}{2}\begin{bmatrix}\mathbf{0} & \dfrac{\dot\lambda(t)}{\lambda^2(t)k}\mathbf{K_{Df}}^{-1}\mathbf{J} \\ \dfrac{\dot\lambda(t)}{\lambda^2(t)k}\mathbf{K_{Df}}^{-1}\mathbf{J} & -d(\mathbf{K_2}^{-1})/dt\end{bmatrix} \tag{39}$$

which is symmetric and lower triangular, and hence, clearly negative definite, so one can write

$$\bar y^T[\mathbf{A_f}^T\mathbf{S} + \mathbf{S}\,\mathbf{A_f}]\bar y + \bar y^T\dot{\mathbf{S}}\,\bar y \le -\underline{\lambda}(\mathbf{W})\|\bar y\|^2 \tag{40}$$

now if the condition (34) is satisfied the matrix $\mathbf{W}$ will be positive definite so from the above equation $\dot V_f$ is negative. Thus V_f is a Lyapunov function and the stability is guaranteed. ■

By this means, we can deduce that the supervisory loop will not essentially affect the stability results previously presented at [15]. It only imposes an adjustment on the stability conditions as in Equation (38). With this fact in mind, in the next subsection we will study the stability of the overall system.

B. Preliminary lemmas for stability analysis

To prove the robust stability of the closed loop system in presence of modeling uncertainty, the Lyapunov direct method is used. Let V be the Lyapunov function candidate as follows

$$V(\bar x, \bar y) = \bar x^T\mathbf{P}\,\bar x + \bar y^T\mathbf{S}\,\bar y \tag{41}$$

in which $\mathbf{S}$ is defined as before (equation (30)) and $\mathbf{P}$ is chosen to be

$$\mathbf{P} = \tfrac{1}{2}\begin{bmatrix}\alpha_2\mathbf{K_P} + \alpha_1\mathbf{K_I} + \alpha_2^2\mathbf{M} & \alpha_2\mathbf{K_D} + \mathbf{K_I} + \alpha_1\alpha_2\mathbf{M} & \alpha_2\mathbf{M} \\ \alpha_2\mathbf{K_D} + \mathbf{K_I} + \alpha_1\alpha_2\mathbf{M} & \alpha_1\mathbf{K_D} + \mathbf{K_P} + \alpha_1^2\mathbf{M} & \alpha_1\mathbf{M} \\ \alpha_2\mathbf{M} & \alpha_1\mathbf{M} & \mathbf{M}\end{bmatrix} \tag{42}$$

in which α_is are real positive constants. The above function in Equation (41) has a quadratic form and it is positive definite due to positive definiteness of $\mathbf{P}$ and $\mathbf{S}$. Positive definiteness of $\mathbf{S}$ has been shown in lemma 1 and the following lemma guarantees that $\mathbf{P}$ is also positive definite in the presence of modeling uncertainty.

Lemma 2: *The matrix* $\mathbf{P}$ *is positive definite if*

$$\alpha_1 > 0\ , \ \alpha_2 > 0\ , \ \alpha_1 + \alpha_2 < 1 \tag{43}$$

$$\alpha_2(k_P - k_D) - (1-\alpha_1)k_I - \alpha_2(1 + \alpha_1 - \alpha_2)\bar m > 0 \tag{44}$$

$$k_P + (\alpha_1 - \alpha_2)k_D - k_I - \alpha_1(1 + \alpha_2 - \alpha_1)\bar m > 0 \tag{45}$$

in which

$$\mathbf{K_P} = k_P\mathbf{I}, \quad \mathbf{K_I} = k_I\mathbf{I}, \quad \mathbf{K_D} = k_D\mathbf{I} \tag{46}$$

Proof is given in [16]. ■

Now for the stability analysis Differentiate V along trajectories (24) and (25), which yields to

$$\begin{aligned}\dot V(\bar x, \bar y) &= \bar x^T\mathbf{P}\,\dot{\bar x} + \dot{\bar x}^T\mathbf{P}\,\bar x + \bar x^T\dot{\mathbf{P}}\,\bar x + \dot{\bar y}^T\mathbf{S}\,\bar y + \bar y^T\mathbf{S}\,\dot{\bar y} + \bar y^T\dot{\mathbf{S}}\,\bar y \\ &= \bar x^T[\mathbf{PA} + \mathbf{A}^T\mathbf{P}]\bar x + \bar x^T\mathbf{B}\Delta\mathbf{A} + \bar x^T\dot{\mathbf{P}}\,\bar x \\ &\quad + 2\bar x^T\mathbf{PC}[\mathbf{I}\quad \mathbf{0}]\bar y + \bar y^T[\mathbf{A_f}^T\mathbf{S} + \mathbf{S}\,\mathbf{A_f}]\bar y + \bar y^T\dot{\mathbf{S}}\,\bar y\end{aligned} \tag{47}$$

the following lemmas will be used to prove that V is a Lyapunov function.

Lemma 3: *For the matrices* $\mathbf{P}$, $\mathbf{A}$, $\mathbf{B}$ *and* $\Delta\mathbf{A}$ *defined previously, the following inequality holds*

$$\bar x^T[\mathbf{PA} + \mathbf{A}^T\mathbf{P}]\bar x + \bar x^T\mathbf{B}\Delta\mathbf{A} + \bar x^T\dot{\mathbf{P}}\,\bar x \le \|\bar x\|(\varepsilon_0 - \varepsilon_1\|\bar x\| + \varepsilon_2\|\bar x\|^2) \tag{48}$$

In this inequality ε_0, ε_1 and ε_2 are real positive constants that depend only on α_1, α_2 and the uncertainty bounds introduced in equations (4) to (6) as follows:

$$\varepsilon_0 = \lambda_1(\beta_0 + \overline{m}\lambda_3) \tag{49}$$

$$\varepsilon_1 = \gamma - \lambda_1\beta_3 - \overline{m}\lambda_2 - \lambda_1\beta_1 \tag{50}$$

$$\varepsilon_2 = \lambda_1\beta_4 + \lambda_1\beta_2 \tag{51}$$

in which

$$\begin{aligned} &\lambda_1 = \overline{\lambda}(\mathbf{R}_1),\ \lambda_2 = \overline{\lambda}(\mathbf{R}_2) \\ &\lambda_3 = \|\ddot{q}_d(t)\|_\infty,\quad \gamma = \underline{\lambda}(\mathbf{Q}) \end{aligned} \tag{52}$$

where

$$\mathbf{Q} = \begin{bmatrix} \alpha_2 k_I \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\alpha_1 k_P - \alpha_2 k_D - k_I)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & k_D \mathbf{I} \end{bmatrix} \tag{53}$$

$$\mathbf{R}_1 = \begin{bmatrix} \alpha_2^2\mathbf{I} & \alpha_1\alpha_2\mathbf{I} & \alpha_2\mathbf{I} \\ \alpha_1\alpha_2\mathbf{I} & \alpha_1^2\mathbf{I} & \alpha_1\mathbf{I} \\ \alpha_2\mathbf{I} & \alpha_1\mathbf{I} & \mathbf{I} \end{bmatrix} \tag{54}$$

$$\mathbf{R}_2 = \begin{bmatrix} \mathbf{0} & \alpha_2^2\mathbf{I} & \alpha_1\alpha_2\mathbf{I} \\ \alpha_2\mathbf{I} & 2\alpha_1\alpha_2\mathbf{I} & (\alpha_1^2+\alpha_2)\mathbf{I} \\ \alpha_1\alpha_2\mathbf{I} & (\alpha_1^2+\alpha_2)\mathbf{I} & \alpha_1\mathbf{I} \end{bmatrix} \tag{55}$$

Proof: Consider the first two terms of the left hand side of (48)

$$\vec{x}^T[\mathbf{PA}+\mathbf{A}^T\mathbf{P}]\ \vec{x} + \vec{x}^T\mathbf{PB}\Delta\mathbf{A} =$$

$$\vec{x}^T[-\mathbf{Q}+\mathbf{R}_2\begin{bmatrix} \mathbf{M} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{M} \end{bmatrix}]\ \vec{x} + \vec{x}^T\begin{bmatrix} \alpha_2\mathbf{I} \\ \alpha_1\mathbf{I} \\ \mathbf{I} \end{bmatrix}\Delta\mathbf{A} \tag{56}$$

in which matrices $\mathbf{R}_2$ and $\mathbf{Q}$ has been defined in Equations (54) and (55). Thus

$$\vec{x}^T[\mathbf{PA}+\mathbf{A}^T\mathbf{P}]\vec{x} + \vec{x}^T\mathbf{PB}\Delta\mathbf{A} \le (-\gamma + \lambda_2\overline{m})\|\vec{x}\|^2 + \lambda_1\|\vec{x}\|\|\Delta\mathbf{A}\| \tag{57}$$

where parameters γ, λ_1 and λ_2 has been introduced in (52). Now consider the last term in (48)

$$\vec{x}^T\dot{\mathbf{P}}\vec{x} = \frac{1}{2}\vec{x}^T\begin{bmatrix} \alpha_2\mathbf{I} \\ \alpha_1\mathbf{I} \\ \mathbf{I} \end{bmatrix}\dot{\mathbf{M}}[\alpha_2\mathbf{I}\quad \alpha_1\mathbf{I}\quad \mathbf{I}]\vec{x} \tag{58}$$

taking into account the fact that for robot manipulators the following equation holds for any vector $\vec{v}$ [17]

$$\vec{v}^T\dot{\mathbf{M}}(\vec{q})\vec{v} = 2\vec{v}^T\mathbf{V_m}(\vec{q},\dot{\vec{q}})\vec{v} \tag{59}$$

equation (58) can be changed to

$$\vec{x}^T\dot{\mathbf{P}}\vec{x} = \vec{x}^T\begin{bmatrix} \alpha_2\mathbf{I} \\ \alpha_1\mathbf{I} \\ \mathbf{I} \end{bmatrix}\mathbf{V_m}[\alpha_2\mathbf{I}\quad \alpha_1\mathbf{I}\quad \mathbf{I}]\vec{x} \tag{60}$$

which yields to

$$\vec{x}^T\dot{\mathbf{P}}\vec{x} \le \lambda_1\|\vec{x}\|^2\|\mathbf{V_m}\| \tag{61}$$

adding Equations (57) and (61) and considering the uncertainty bounds (4) and (5) ends the proof.

■

Lemma 4: *For the matrix* **C** *defined previously the following inequality holds*

$$2\vec{x}^T\mathbf{PC}[\mathbf{I}\quad \mathbf{0}]\vec{y} \le 2\|\vec{x}\|\overline{\lambda}(\mathbf{P})\overline{\lambda}(\mathbf{M}^{-1})\|\vec{y}\| \tag{62}$$

Proof: Considering that

$$\mathbf{C}[\mathbf{I}\quad \mathbf{0}] = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{M}^{-1}(\vec{q}) & \mathbf{0} \end{bmatrix} \tag{63}$$

proof is straightforward. ■

Lemma 5: *Suppose that the Lyapunov function of a dynamic system has the following properties*

$$\dot{V}(X) \le \|X\|(\phi_0 - \phi_1\|X\| + \phi_2\|X\|^2) \tag{64}$$

and

$$\underline{\lambda}\|X\|^2 \le V(X) \le \overline{\lambda}\|X\|^2 \tag{65}$$

where $\underline{\lambda}, \overline{\lambda}$ *and* ϕ_i*s are constants. Given that*

$$d = \frac{2\phi_0}{\phi_1 + \sqrt{\phi_1^2 - 4\phi_0\phi_2}} \times \sqrt{\frac{\overline{\lambda}}{\underline{\lambda}}} \tag{66}$$

then the system is UUB stable with respect to $\mathbf{B}(0,d)$*, provided that*

$$\phi_1 > 2\sqrt{\phi_0\phi_2} \tag{67}$$

$$\phi_1\left[\phi_1 + \sqrt{\phi_1^2 - 4\phi_0\phi_2}\right] > 2\phi_0\phi_2\left(1 + \sqrt{\frac{\overline{\lambda}}{\underline{\lambda}}}\right) \tag{68}$$

$$\phi_1 + \sqrt{\phi_1^2 - 4\phi_0\phi_2} > 2\phi_2\|X_0\|\sqrt{\frac{\overline{\lambda}}{\underline{\lambda}}} \tag{69}$$

where $\|X_0\|$ *denotes the initial condition.*

Proof can be found in [10] under proof of lemma 3.5 in it.

C. Stability of the complete system

In this subsection we present the main result. To be compact we simply refer to the equations by their numbers in the body of the theorem.

Theorem 2: *Consider the flexible joint manipulator of equations (15) and (16) with the composite controller structure of equations (17) and (18), under supervisory loop. The overall closed loop system with governing equations of motion (24) and (25) is UUB stable and the state variables converge to the origin under conditions of theorem 1 and lemma 2 and some new certain limits imposed on the fast (PD) and slow (PID) controller gains which will be deduced at the end of the following proof.*

Proof: The Lyapunov function candidate V introduced in (41) has been shown to be positive definite. This imposes conditions of lemma 2, equations (43) to (45), to be satisfied. Now in order to study the negativity of $\dot{V}(\vec{x},\vec{y})$, consider (47),(48), (62) and (40) which yield

$$\begin{aligned} \dot{V}(\vec{x},\vec{y}) \le\ & \|\vec{x}\|(\varepsilon_0 - \varepsilon_1\|\vec{x}\| + \varepsilon_2\|\vec{x}\|^2) \\ & + 2\|\vec{x}\|\overline{\lambda}(\mathbf{P})\overline{\lambda}(\mathbf{M}^{-1})\|\vec{y}\| - \overline{\lambda}(\mathbf{W})\|\vec{y}\|^2 \\ & = [\|\vec{x}\|\quad \|\vec{y}\|]\begin{bmatrix} -\varepsilon_1 & \overline{\lambda}(\mathbf{P})\overline{\lambda}(\mathbf{M}^{-1}) \\ \overline{\lambda}(\mathbf{P})\overline{\lambda}(\mathbf{M}^{-1}) & -\overline{\lambda}(\mathbf{W}) \end{bmatrix}\begin{bmatrix} \|\vec{x}\| \\ \|\vec{y}\| \end{bmatrix} \\ & + \varepsilon_0\|\vec{x}\| + \varepsilon_2\|\vec{x}\|^3 \end{aligned} \tag{70}$$

or

$$\dot{V}(\vec{x},\vec{y}) \le -\vec{z}_1^T\mathbf{R}\vec{z}_1 + \varepsilon_0\|\vec{x}\| + \varepsilon_2\|\vec{x}\|^3 \tag{71}$$

in which

$$\vec{z}_1 = \begin{bmatrix} \|\vec{x}\| \\ \|\vec{y}\| \end{bmatrix},\ \mathbf{R} = -\begin{bmatrix} -\varepsilon_1 & \overline{\lambda}(\mathbf{P})\overline{\lambda}(\mathbf{M}^{-1}) \\ \overline{\lambda}(\mathbf{P})\overline{\lambda}(\mathbf{M}^{-1}) & -\overline{\lambda}(\mathbf{W}) \end{bmatrix} \tag{72}$$

here we see that the matrix **W** must be positive definite thus condition (33) or (34) must be satisfied, as well. Now if we define

$$\bar{z} = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \tag{73}$$

we have

$$\|\bar{z}\| = \|\bar{z}_1\|, \ \|\bar{z}\| \geq \|\bar{x}\| \tag{74}$$

thus

$$\dot{V}(\bar{x}, \bar{y}) \ \leq \ \|\bar{z}\|(\varepsilon_0 - \bar{\lambda}(\mathbf{R})\|\bar{z}\| + \varepsilon_2 \|\bar{z}\|^2) \tag{75}$$

Now apply the Riley Ritz inequality which reads

$$\begin{aligned} &\underline{\lambda}(\mathbf{P})\|\bar{x}\|^2 \leq \bar{x}^T \mathbf{P}\, \bar{x} \leq \bar{\lambda}(\mathbf{P})\|\bar{x}\|^2 \\ &\underline{\lambda}(\mathbf{S})\|\bar{y}\|^2 \leq \ \bar{y}^T \mathbf{S}\, \bar{y} \leq \bar{\lambda}(\mathbf{S})\|\bar{y}\|^2 \end{aligned} \tag{76}$$

adding these two equations yields

$$\bar{z}_1^{\,T} \begin{bmatrix} \underline{\lambda}(\mathbf{P}) & 0 \\ 0 & \underline{\lambda}(\mathbf{S}) \end{bmatrix} \bar{z}_1 \leq V(\bar{z}) \leq \bar{z}_1^{\,T} \begin{bmatrix} \bar{\lambda}(\mathbf{P}) & 0 \\ 0 & \bar{\lambda}(\mathbf{S}) \end{bmatrix} \bar{z}_1 \tag{77}$$

in other words

$$\underline{\lambda}\|\bar{z}\|^2 \leq V(\bar{z}) \leq \bar{\lambda}\,\|\bar{z}\|^2 \tag{78}$$

in which

$$\begin{aligned} &\bar{\lambda} = \max\{\bar{\lambda}(\mathbf{P}), \bar{\lambda}(\mathbf{S})\} \\ &\underline{\lambda} = \min\{\underline{\lambda}(\mathbf{P}), \underline{\lambda}(\mathbf{S})\} \end{aligned} \tag{79}$$

Now from (69) and (72) and by lemma 5 we can state that given that

$$d = \frac{2\varepsilon_0}{\bar{\lambda}(\mathbf{R}) + \sqrt{\bar{\lambda}(\mathbf{R})^2 - 4\varepsilon_0\varepsilon_2}} \times \sqrt{\frac{\bar{\lambda}}{\underline{\lambda}}} \tag{80}$$

the system is UUB stable with respect to $\mathbf{B}(0,d)$, provided the following stability conditions are satisfied

$$\bar{\lambda}(\mathbf{R}) > 2\sqrt{\varepsilon_0\varepsilon_2} \tag{81}$$

$$\bar{\lambda}(\mathbf{R})\left[\bar{\lambda}(\mathbf{R}) + \sqrt{\bar{\lambda}(\mathbf{R})^2 - 4\varepsilon_0\varepsilon_2}\right] > 2\varepsilon_0\varepsilon_2\left(1 + \sqrt{\frac{\bar{\lambda}}{\underline{\lambda}}}\right) \tag{82}$$

$$\bar{\lambda}(\mathbf{R}) + \sqrt{\bar{\lambda}(\mathbf{R})^2 - 4\varepsilon_0\varepsilon_2} > 2\varepsilon_2\|z_0\|\sqrt{\frac{\bar{\lambda}}{\underline{\lambda}}} \tag{83}$$

where $\|z_0\|$ denotes the initial condition.

■

This proof reveals an important aspect of the supervisory loop dynamics included in the proposed controller law. Because of the dynamical gain adaptation of the controller, this adaptation preserves the robust stability of the system, without perturbing the stability conditions. Another aspect that can be concluded from this analysis is the robustness property of the stability, in presence of modeling uncertainty. Since the unmodeled but bounded dynamics of the system is systematically encapsulated in the system model (as stated in Equations (3) to (5)), the only influence this will impose on the stability is the respective controller gains bound depicted in the above mentioned conditions.

V. Conclusions

In this paper the problem of controller synthesis for flexible joint robots in presence of actuator saturation is analyzed in detail. The singularly perturbed model of the system is first introduced briefly, and a composite controller structure is proposed for the system. In order to remedy the limitations caused by actuator bounds, a supervisory loop is proposed, and it is shown that a model free fuzzy supervisory loop makes it possible to preserve stability, without great loss in performance. The supervisor will affect the signals *in prior to* the controller, and therefore, affecting the controller states. This is on contrary to the static saturation block which will be placed *after* the controller. It is shown through a Lyapunov based stability analysis, that due to the structure of the supervisory loop, and regardless of the logic it uses, since the controller adaptation gain is bounded and the overall variation of the system energy is dissipative, the stability condition of the composite controller remains unchanged. The detail stability analysis of the overall closed loop system is performed using Lyapunov direct method and the robust stability conditions are derived, respectively. These all considerations have enabled us to offer an implementable controller with guaranteed robust stability which is an essential requirement for susceptible applications such as space robotics.

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