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Robust H_∞ Filtering for Nonlinear Uncertain Systems Using State-Dependent Riccati Equation Technique

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***Abstract*—The standard state-dependent Riccati equation (SDRE) filter, which is set up by direct SDC parameterization, demands complete knowledge of the system model, and the disturbance inputs characteristics. However, this inherent dependency can severely degrade its performance in practical applications. In this paper, based on the H_∞ norm minimization criterion, a robust SDRE filter is proposed to effectively estimate the states of nonlinear uncertain systems exposed to unknown disturbance inputs. Considering a Lipschitz condition on the chosen SDC form, we guarantee fulfillment of a modified H_∞ performance index by the proposed filter. The effectiveness of the robust SDRE filter is demonstrated through numerical simulations where it brilliantly outperforms the usual SDRE filters in presence of model uncertainties as well as process and measurement noises.**

I. Introduction

OVER the past decade, state-dependent Riccati equation (SDRE) filtering techniques have been extensively used for nonlinear state/parameter estimation within aerospace and power electronics applications [1]-[4]. Unlike the broadly acceptable extended Kalman filter (EKF) [5], the SDRE filter does not involve the Jacobian evaluations, but it entails direct parameterization [2], [6]. Briefly, it fully captures the nonlinearities of the system and brings the nonlinear system into a nonunique linear structure having state-dependent coefficients (SDCs). This nonuniqueness of the SDC form provides design flexibility which can be exploited to overcome serious difficulties such as singularities and loss of observability in traditional filtering methods [6].

Generally, there are two commonly used approaches for the SDRE filtering technique. The first approach, proposed originally by Mracek *et al.* in [6], is essentially constructed by considering the dual problem of the well-known SDRE nonlinear control law. The resulting filter has the same structure as the steady-state linear Kalman filter and the Kalman gain is obtained by solving a state dependent algebraic Riccati equation (SDARE) [7]. However, as reported in [8], this solution may be computationally expensive for large scale systems and depends significantly on the observability property of the system.

The other approach is recently suggested in the literature [7]-[9], and has the same structure as the linear Kalman filter. Indeed, it removes the infinite time horizon assumption and requires the integration of a state-dependent differential Riccati equation (SDDRE) [7]. This alternative approach addresses the issues of high computational load and the restrictive observability requirement in the algebraic form of the estimator.

Although the practical usefulness of the SDRE filter has been demonstrated through impressive simulation results, rigorous mathematical investigations of the filter have been presented very recently [2], [8]-[11]. Assuming certain observability and Lipschitz conditions on the SDC factorization and considering an incremental splitting of the state-dependent matrices, the local convergence of the continuous-time algebraic SDRE observer is proven in [10]. It is also shown in [2] how this observer converges asymptotically to the first-order minimum variance estimate given by the EKF. The analysis is based on stable manifold theory and Hamilton-Jacobi-Bellman (HJB) equations. Moreover, the analogous discrete-time difference observer is treated in [8] and [9], where two distinct sufficient conditions sets for its asymptotic stability are provided. The authors have also modified the differential SDRE observer in order to obtain an exponential observer with a noticeable superior performance and an increased region of attraction [11].

All the theoretical results cited above are confined to the nonlinear deterministic processes and assume that the system model is *perfectly* known. Applying the standard SDRE filters to general stochastic systems, that is inevitable in practical purposes, requires accurate specification of the noise statistics as well. However, model uncertainty and incomplete statistical information are often encountered in real applications which may potentially give rise to excessive estimation errors. To tackle such difficulties, we propose a robust SDRE filter with guaranteed H_∞ performance criterion. The motivation of this paper stems from the fact that in contrast to successful derivation of an H_∞ formulation concerning the SDRE control, accomplished by Cloutier *et al.* in [12], there is no documented similar attempt concerning its filtering counterpart.

Since the pioneer works of linear H_∞ filtering designs (see, e.g., the celebrated papers [13], [14]), the nonlinear H_∞ filtering problem has been studied by a number of authors

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(see [15], [16] for a basic study, [17] for a general stochastic investigation, [18] for a fuzzy investigation, and [19]-[21] for approximate solutions). In this paper, we consider a general continuous-time nonlinear uncertain model as represented by Nguang and Fu, [15], to develop a robust H_∞ SDRE-based filter for nonlinear uncertain systems exposed to additive disturbance inputs. The proposed filter involves neither solving the Hamilton-Jacobi inequalities (HJIs) in [15]-[17], nor constructing exact Takagi-Sugeno fuzzy models in [18], which are both time-consuming hard tasks except for some special cases. In addition, it obviates the need for linearization procedure of the extended H_∞ techniques [19], [20] and the Riccati-based filtering design [21], and exhibits robustness against not only unknown disturbances but also model uncertainties.

Precisely speaking, we intend to robustify the standard differential SDRE filter such that the estimation error dynamics is norm-bounded and achieves a prescribed level of disturbance attenuation for all admissible uncertainties. The key assumption made is that the SDC parameterization is chosen so that the state-dependent matrices are Lipschitz, at least locally. Note that our result is substantially different from the well-established methods associated with the Lipschitz nonlinear systems, which decompose the entire model into a linear unforced part and a Lipschitz nonlinear uncertain part (cf. [22] and the references quoted therein). In comparison with the algorithms in [15]-[18], another advantage of the proposed method is its simplicity, as no complicated computation procedures are required to implement the estimator. It can be implemented systematically and inherits the elaborated capabilities of the SDRE-based filters [23], as well.

The rest of the paper is organized as follows. Section II gives the necessary backgrounds and formulates the uncertain SDC description along with the robust SDRE filter. In Section III, by employing an appropriate Lyapunov function we derive the performance index for the proposed filter, which can be regarded as a modification of the conventional H_∞ performance criterion (cf. [15], [16]). Section IV provides a simulation example to illustrate some definite superiority of the proposed filter over the corresponding usual SDRE-based filters. Finally, some conclusions are drawn in Section V.

I. Robust SDRE Filter And Preliminaries

Consider a smooth nonlinear uncertain system described by continuous-time equations of the following form (parameter t is omitted in trivial cases for notational convenience)

$$\dot{x}(t) = f(x) + \Delta f(x) + G(t)w_0(t) \qquad (2.1)$$

$$y(t) = h(x) + \Delta h(x) + D(t)v_0(t) \qquad (2.2)$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^m$ is the measured output, and $w_0(t) \in \mathbb{R}^p$ and $v_0(t) \in \mathbb{R}^q$, which stand for exogenous disturbance inputs, are process and measurement noises with unknown statistical properties. For the sake of simplicity, we are restricted to unforced noise-driven systems, a slightly more general representation than that of [15]. Some remarks on the forced case affine in the control are given in Section III. The nonlinear system dynamic $f(x)$ and the observation model $h(x)$ are assumed to be known C^1-functions. $G(t)$ and $D(t)$ are time varying known matrices of size $n \times p$ and $m \times q$, respectively. Also, $\Delta f(x)$ and $\Delta h(x)$ represent the system model uncertainties.

Assumption 2.1: Let the model uncertainties satisfy

$$\begin{bmatrix} \Delta f(x) \\ \Delta h(x) \end{bmatrix} = \begin{bmatrix} E_1(t)\Delta_1(t) \\ E_2(t)\Delta_2(t) \end{bmatrix} N(x) \qquad (2.3)$$

in which $N(x) \in C^1$, $E_1(t)$ and $E_2(t)$ are known matrix functions with appropriate dimensions that characterize the structure of the uncertainties. Also, $\Delta_1(t)$ and $\Delta_2(t)$ are norm-bounded unknown matrices.

By performing direct parameterization, the nonlinear dynamics (2.1) and (2.2) accompanied by Assumption 2.1 can be put into the following *uncertain* state-dependent coefficient (SDC) form

$$\dot{x}(t) = A(x)x + E_1(t)\Delta_1(t)N(x) + G(t)w_0(t) \qquad (2.4)$$

$$y(t) = C(x)x + E_2(t)\Delta_2(t)N(x) + D(t)v_0(t) \qquad (2.5)$$

where $f(x) = A(x)x$ and $h(x) = C(x)x$. Note that the SDC parameterization is unique only if x is scalar [12] (also see the remark given below). Besides, the smoothness of the vector functions $f(x)$ and $h(x)$ with $f(0) = h(0) = 0$ makes it feasible [6], [12] (see also [23] for effective handling of situations which prevent a straightforward parameterization).

Remark 2.1: If $A_1(x)$ and $A_2(x)$ are two distinct factorization of $f(x)$, then

$$A_3(x) = M(x)A_1(x) + \left(I - M(x)\right)A_2(x)$$

is also a parameterization of $f(x)$ for each matrix-valued function $M(x) \in \mathbb{R}^{n\times n}$. This is an exclusive characteristic of all SDRE-based design techniques, which has been successfully used not only to avoid singularity or loss of observability, but also to enhance performance (cf. [2], [6], [9]). Moreover, it may be used to satisfy the Lipschitz condition in our filtering design (see Remark 3.2).

Let us define the signal to be estimated as follows

$$z(x(t)) \triangleq L(t)x(t) \qquad (2.6)$$

where $z(x) \in \mathbb{R}^s$ can be viewed as the filter output, and $L(t)$ is a known $s \times n$ matrix bounded via

$$\underline{l}I \le L^T(t)L(t) \le \overline{l}I \quad (2.7)$$

for every $t \ge 0$ with some positive real numbers $\underline{l}$, $\overline{l}$.

We seek to propose a dynamic filter for the uncertain SDC model, given by (2.4) and (2.5), which robustly estimates the quantity $z(x)$ from the observed data $y(t)$ with a guaranteed H$_\infty$ performance criterion. In other words, it is desired to ensure a bounded energy gain from the input noises ($w_0(t)$, $v_0(t)$) to the estimation error in terms of the H$_\infty$ norm. The proposed filter has an SDRE-like structure and is prescribed to be

$$\begin{aligned}\dot{\hat{x}} = A(\hat{x})\hat{x} + K(t)\left[y(t) - C(\hat{x})\hat{x}\right] \\ + \mu^{-2}P(t)\nabla_x N(\hat{x})^T N(\hat{x})\end{aligned} \quad (2.8)$$

In the above, $\mu > 0$ is a free design parameter and ∇_x denotes the gradient with respect to x. $\hat{x}(t)$ represents the estimated state vector and the filter gain matrix, $K(t) \in R^{n\times m}$, is defined as

$$K(t) = P(t)C^T(\hat{x})R^{-1} \quad (2.9)$$

i.e., in the same way as for the usual SDRE filter. The positive definite matrix $P(t)$ is updated through the following state-dependent differential Riccati equation (SDDRE)

$$\begin{aligned}\dot{P}(t) = A(\hat{x})P(t) + P(t)A^T(\hat{x}) + \Gamma(t)Q\Gamma^T(t) \\ - P(t)\Big[C^T(\hat{x})R^{-1}C(\hat{x}) - \mu^{-2}\nabla_x N(\hat{x})^T\nabla_x N(\hat{x}) \\ -\lambda^{-2}\nabla_x z(\hat{x})^T\nabla_x z(\hat{x})\Big]P(t)\end{aligned} \quad (2.10)$$

with

$$\Gamma(t) = \left[\mu E_1(t) \quad G(t)\right], \quad (2.11)$$

positive definite matrix $Q \in \mathbb{R}^{n\times n}$, symmetric positive definite matrix $R \in \mathbb{R}^{m\times m}$, and a given positive real value $\lambda > 0$ that indirectly indicates the level of disturbance attenuation in our robust filter design. This verity as well as the exact role of the free parameter μ will be clarified in the next section.

Remark 2.2: It can be easily seen that with $\lambda, \mu \to \infty$, the proposed filter reverts to the standard differential SDRE filter [7], [11]. Meanwhile, setting $\mu = \infty$ together with replacing $A(\hat{x})$ and $C(\hat{x})$ by the Jacobian of $f(x)$ and $h(x)$, respectively, in (2.9) and (2.10) render the structure of the extended H$_\infty$ filter [19], [20].

Before analyzing the performance of the robust SDRE filter, we recall two preparatory definitions within the H$_\infty$ filtering theory.

Definition 2.1: (Extended L_2-space) The set $L_2[0,T]$ consists of all Lebesque measurable functions $g(t) \in \mathbb{R}^+ \to \mathbb{R}^r$ such that

$$\int_0^T \|g(t)\|^2\, dt < \infty \quad (2.12)$$

for every $T \ge 0$ with $\|g(t)\|$ as the Euclidian norm of the vector $g(t)$ (see, e.g., [16], [22]).

Definition 2.2: (Robust H$_\infty$ SDRE Filtering) Given any real scalar $\gamma > 0$, the dynamic SDRE filter (2.8)-(2.10) associated with the dynamics (2.4)-(2.6) is said to satisfy the H$_\infty$ performance criterion if

$$\int_0^T \|z(t) - \hat{z}(t)\|^2\, dt \le \gamma^2 \int_0^T \left(\|w_0(t)\|_W^2 + \|v_0(t)\|_V^2\right)dt \quad (2.13)$$

holds for all $T \ge 0$, all $v_0(t), w_0(t) \in L_2\left[0,T\right]$, and all admissible uncertainties. Where, $\|w_0(t)\|_W$ and $\|v_0(t)\|_V$ are taken to be Euclidian norms scaled by some positive matrices W and V, respectively.

Remark 2.3: Inequality (2.13) implies that the L_2-gain from the exogenous inputs ($w_0(t)$, $v_0(t)$) to $z(t) - \hat{z}(t)$, called the generalized estimation error, is less than or equal to some minimum value γ^2. It only necessitates that the disturbances have finite energy which is a familiar mild assumption.

Remark 2.4: Definition 2.2 is derived from what was declared by Nguang and Fu, [15]. The difference is that we consider *two* distinct noise sources with *scaled* Euclidian norms. These scalings, which are similar to those introduced in [20], may be interpreted as simple weights because H$_\infty$ filtering does not rely on the availability of statistical information.

II. H$_\infty$ Performance Analysis

In this section, we analyze the estimation error dynamics to derive an interesting feature of the proposed robust filter, which will be properly called modified H$_\infty$ performance index. This criterion reveals the ability of the filter to minimize the effects of disturbances and uncertainties on the estimation error.

In order to facilitate our analysis, we adopt the following notation

$$\begin{aligned}w(t) &= [\mu^{-1}\Delta_1(t)N(x) \quad w_0(t)]^T \\ v(t) &= \left[I \quad I\right][E_2(t)\Delta_2(t)N(x) \quad D(t)v_0(t)]^T\end{aligned} \quad (3.1)$$

in which the uncertainties are treated as some fictitious noises. The estimation error is defined by

$$e(t) = x(t) - \hat{x}(t) \quad (3.2)$$

Subtracting (2.8) from (2.4) and considering (3.1) and (2.11), the error dynamics is expressed as

$$\begin{aligned}\dot{e}(t) = A(x)x + \Gamma(t)w(t) - A(\hat{x})\hat{x} \\ - K(t)\left[y(t) - C(\hat{x})\hat{x}\right] - \mu^{-2}P(t)\nabla_x N(\hat{x})^T N(\hat{x})\end{aligned} \quad (3.3)$$

Adding and subtracting $A(\hat{x})x$ to the whole equation, and adding and subtracting $C(\hat{x})x$ into the bracket lead to

$$\begin{aligned}\dot{e}(t) &= A(\hat{x})(x-\hat{x}) + \left(A(x)-A(\hat{x})\right)x \\ &\quad - K(t)\left[C(\hat{x})(x-\hat{x}) + \left(C(x)-C(\hat{x})\right)x + v(t)\right] \\ &\quad + \Gamma(t)w(t) - \mu^{-2}P(t)\nabla_x N(\hat{x})^T N(\hat{x})\end{aligned} \quad (3.4)$$

rearranging the terms together with (3.2), we have

$$\begin{aligned}\dot{e}(t) &= \left[A(\hat{x}) - K(t)C(\hat{x})\right]e(t) + \alpha(x,\hat{x}) - K(t)\beta(x,\hat{x}) \\ &\quad - K(t)v(t) + \Gamma(t)w(t) - \mu^{-2}P(t)\nabla_x N(\hat{x})^T N(\hat{x})\end{aligned} \quad (3.5)$$

where the nonlinear functions $\alpha(x,\hat{x})$ and $\beta(x,\hat{x})$ are given by

$$\alpha(x,\hat{x}) = \left[A(x) - A(\hat{x})\right]x \quad (3.6)$$

$$\beta(x,\hat{x}) = \left[C(x) - C(\hat{x})\right]x \quad (3.7)$$

The requirements for Theorem 3.1 given below, which embodies the main result of this paper, are summarized by the following assumptions.

Assumption 3.1: The state-dependent matrix $C(x)$ and the state vector $x(t)$ are bounded via

$$\left\|C(x)\right\| \le \bar{c} \quad (3.8)$$

$$\left\|x(t)\right\| \le \sigma \quad (3.9)$$

for all $t \ge 0$ and some positive real numbers $\bar{c}, \sigma > 0$.

Remark 3.1: Note that the assumption above is not severe. In particular, for many applications boundedness of the state variables, which often represent physical quantities, is natural. Besides, if $C(x)$ fulfill (3.8) for every physical reasonable value of the state vector $x(t)$, we may suppose without loss of generality that (3.8) holds.

Assumption 3.2: The SDC parameterization is chosen such that $A(x)$ and $C(x)$ are at least locally Lipschitz, i.e., there exist constants $k_A, k_C > 0$ such that

$$\left\|A(x) - A(\hat{x})\right\| \le k_A \left\|x - \hat{x}\right\| \quad (3.10)$$

$$\left\|C(x) - C(\hat{x})\right\| \le k_C \left\|x - \hat{x}\right\| \quad (3.11)$$

hold for all $x, \hat{x} \in \mathbb{R}^n$ with $\left\|x-\hat{x}\right\| \le \varepsilon_A$ and $\left\|x-\hat{x}\right\| \le \varepsilon_C$, respectively.

It should be mentioned that if the SDC form fulfills the Lipschitz condition globally in $\mathbb{R}^n$, then all the results in this section will be valid globally.

Remark 3.2: Inequalities (3.10)-(3.11) are the key conditions in our performance analysis. They are similar to Lipschitz conditions imposed in [10] and [11], and may be difficult to satisfy for some nonlinear dynamics. Nevertheless, additional degrees of freedom provided by nonuniqueness of the SDC parameterization can be exploited to realize Assumption 3.2.

With these prerequisites we are able to state the following theorem, which demonstrates how a modified H_∞ performance index is met by applying the proposed SDRE filter.

Theorem 3.1: Consider the nonlinear uncertain system of (2.4)-(2.6) along with the robust SDRE filter described by (2.8)-(2.10) with some $\lambda, \mu > 0$ and positive definite matrices Q and R. Under assumptions 3.1-3.2, the generalized estimation error $z(t) - \hat{z}(t)$ fulfills a modified type of the H_∞ performance criterion introduced in Definition 2.2, provided that the SDDRE (2.10) has a positive definite solution for all $t \ge 0$ and λ is chosen such that

$$\lambda^{-2}\underline{l} > 2\kappa \quad (3.12)$$

where

$$\kappa = \left(\frac{k_A}{\underline{p}} + \frac{\bar{c}k_C}{\underline{r}}\right)\sigma \quad (3.13)$$

and $\underline{r}, \underline{p} > 0$ denote the smallest eigenvalue of the positive definite matrices R and $P(t)$, respectively. Furthermore, the disturbance attenuation level, γ in (2.13), is given by

$$\gamma^2 = \frac{\bar{l}}{\lambda^{-2}\underline{l} - 2\kappa} \quad (3.14)$$

with $\bar{l}, \underline{l}$ in (2.7).

Remarks:

3.3) For the usual differential SDRE filter, the solution of the standard SDDRE is positive definite and also bounded above if the SDC form satisfies a certain uniform detectability condition as stated in [11] (cf. [8], [9] for similar relation to observability condition in the corresponding discrete-time filter). Unfortunately, this condition cannot be applied to get similar results for the H_∞-filtering-like SDDRE (2.10). However, it is a well-known problem arising in H_∞ control as well as H_∞ filtering that the solutions of the related Riccati equations may lack being positive definite (cf. [21]).

3.4) The existence of a positive definite solution $P(\cdot)$ for the SDDRE (2.10) depends mainly on an appropriate choice of λ and μ. To find suitable values for λ and μ one can employ a binary search algorithm, which is widely used to solve H_∞ control and H_∞ filtering problems (see, e.g., [21], [22])

3.5) Clearly, the filter attenuation constant γ is indirectly specified by the design parameter λ while it is independent of μ. The extra design parameter μ has turned out to be very useful for ensuring solvability of (2.10) with the desired positive definiteness property. Further, it scales the uncertainty norm in our performance index (see the proof of Theorem 3.1 given below).

3.6) Inequality (3.12) roughly means that λ is chosen sufficiently small. Surprisingly, this is in accordance with the purpose of performance improvement which calls for a small value of γ in (3.14). Also see the following remark.

3.7) It can be shown that, (3.12) is obviated while the estimation error still assures the same performance index with different attenuation constant $\gamma^2 = \lambda^2\left(2\overline{l}/\underline{l}\right)$, if inequalities (3.10)-(3.11) are replaced by more restricted Lipschitz conditions with two exponent, e.g., $\|A(x) - A(\hat{x})\| \le k_A \|x - \hat{x}\|^2$. The proof of theorem can be modified easily for this case.

To prove Theorem 3.1 we need the following preparation.

Lemma 3.1: Let inequalities (3.8)-(3.11) are valid. Then for an estimation error $\|e\| \le \varepsilon$, $\Pi(t) = P(t)^{-1}$ satisfies the inequality

$$(x-\hat{x})^T \Pi(t)\left[\alpha(x,\hat{x}) - K(t)\beta(x,\hat{x})\right] \le \kappa \|x-\hat{x}\|^2 \quad (3.15)$$

where $\varepsilon = \min(\varepsilon_A, \varepsilon_C)$. The positive real scalar κ, the matrix $K(t)$, and the nonlinearities α, β are given by (3.13), (2.9), (3.6), and (3.7) respectively.

Proof: Applying the triangle inequality, $K = PC^T(\hat{x})R^{-1}$ and $\Pi P = I$ leads to

$$\begin{aligned}&\left\|(x-\hat{x})^T \Pi \alpha(x,\hat{x}) - (x-\hat{x})^T \Pi K \beta(x,\hat{x})\right\| \le \\ &\left\|(x-\hat{x})^T \Pi \alpha(x,\hat{x})\right\| + \left\|(x-\hat{x})^T C^T(\hat{x}) R^{-1} \beta(x,\hat{x})\right\|\end{aligned} \quad (3.16)$$

In view of the Lipschitz conditions on the SDC form and inequality (3.9), the nonlinear functions α, β are bounded via

$$\|\alpha(x,\hat{x})\| = \left\|\left[A(x) - A(\hat{x})\right]x\right\| \le k_A \sigma \|x-\hat{x}\| \quad (3.17)$$

$$\|\beta(x,\hat{x})\| = \left\|\left[C(x) - C(\hat{x})\right]x\right\| \le k_C \sigma \|x-\hat{x}\| \quad (3.18)$$

with $\|x-\hat{x}\| \le \varepsilon_A$ and $\|x-\hat{x}\| \le \varepsilon_C$, respectively. Choosing $\varepsilon = \min(\varepsilon_A, \varepsilon_C)$ and employing (3.17), (3.18), (3.8), $\|\Pi\| \le 1/\underline{p}$, and $\|R^{-1}\| \le 1/\underline{r}$ in (3.16), we obtain

$$\begin{aligned}&\left\|(x-\hat{x})^T \Pi \alpha(x,\hat{x},u) - (x-\hat{x})^T \Pi K \beta(x,\hat{x})\right\| \le \\ &\|x-\hat{x}\| \frac{k_A \sigma}{\underline{p}} \|x-\hat{x}\| + \|x-\hat{x}\| \frac{\overline{c} k_C \sigma}{\underline{r}} \|x-\hat{x}\|\end{aligned} \quad (3.19)$$

therefore (3.15) follows immediately with κ given in (3.13). ■

Proof of Theorem 3.1: Choose a Lyapunov function as follows

$$V(e(t)) = e^T(t)\Pi(t)e(t) \quad (3.20)$$

with $\Pi(t) = P(t)^{-1}$, which exists since $P(t)$ in (2.10) is supposed to be positive definite. Taking time derivative of $V(e)$ we get

$$\begin{aligned}\dot{V}(e(t)) = &\dot{e}^T(t)\Pi(t)e(t) + e^T(t)\dot{\Pi}(t)e(t) \\ &+ e^T(t)\Pi(t)\dot{e}(t)\end{aligned} \quad (3.21)$$

Inserting (3.5) and (2.10) in (3.21) along with considering $\dot{\Pi}(t) = -\Pi(t)\dot{P}(t)\Pi(t)$, yield with a few rearrangement

$$\begin{aligned}\dot{V}(e(t)) = &e^T\left[-\lambda^{-2}L^T L\right]e + 2e^T \Pi\left[\alpha - K\beta\right] \\ &+ w^T \Gamma^T \Pi e + e^T \Pi \Gamma w - v^T R^{-1} C(\hat{x}) e \\ &- e^T C^T(\hat{x}) R^{-1} v - e^T C^T(\hat{x}) R^{-1} C(\hat{x}) e \\ &- e^T \Pi \Gamma Q \lambda \Gamma^T \Pi e \\ &+ \mu^{-2}\Big\{-e^T\left(\nabla_x N(\hat{x})^T \nabla_x N(\hat{x})\right)e \\ &- e^T \nabla_x N(\hat{x})^T N(\hat{x}) - \left(\nabla_x N(\hat{x})^T N(\hat{x})\right)^T e\Big\}\end{aligned} \quad (3.22)$$

Let us set $s = Q^{-(1/2)}w - (\Gamma Q^{(1/2)})^T \Pi e$ and $\eta = v + C(\hat{x})e$, then (3.22) can be rewritten as

$$\begin{aligned}\dot{V}(e(t)) = &e^T\left[-\lambda^{-2}L^T L\right]e + 2e^T \Pi\left[\alpha - K\beta\right] \\ &+ w^T Q^{-1} w - s^T s + v^T R^{-1} v - \eta^T R^{-1} \eta + \\ &\mu^{-2}\Big\{-e^T\left(\nabla_x N(\hat{x})^T \nabla_x N(\hat{x})\right)e \\ &- e^T \nabla_x N(\hat{x})^T N(\hat{x}) - \left(\nabla_x N(\hat{x})^T N(\hat{x})\right)^T e\Big\}\end{aligned} \quad (3.23)$$

where $Q = Q^{(1/2)}Q^{(1/2)T}$, $R = R^{(1/2)}R^{(1/2)T}$. Completing the square in the accolade of (3.23) and utilizing the triangle inequality, we obtain by virtue of Lemma 3.1

$$\begin{aligned}\dot{V}(e(t)) \le &e^T\left[-\lambda^{-2}L^T L\right]e + 2\kappa\|e\|^2 \\ &+ w^T Q^{-1} w + v^T R^{-1} v + \mu^{-2} N^T(\hat{x}) N(\hat{x})\end{aligned} \quad (3.24)$$

provided that the estimation errors satisfy $\|e\| \le \varepsilon$, where $\varepsilon = \min(\varepsilon_A, \varepsilon_C)$. The use of $-\overline{l}I \le -L^T L \le \underline{l}I$ leads to

$$\begin{aligned}\dot{V}(e(t)) \le &-\frac{\lambda^{-2}\underline{l} - 2\kappa}{\overline{l}} e^T\left[L^T L\right]e \\ &+ w^T Q^{-1} w + v^T R^{-1} v + \mu^{-2} N^T(\hat{x}) N(\hat{x})\end{aligned} \quad (3.25)$$

By integrating both sides of (3.25) over the time interval $[0,T]$, the H_∞ performance index of the proposed filter is derived as

$$\begin{aligned}\int_0^T \|L(t)e(t)\|^2 dt \le \gamma^2 \int_0^T \Big(&\|w(t)\|^2_{Q^{-1}} + \|v(t)\|^2_{R^{-1}} \\ &+ \|N(\hat{x})\|^2_{\mu^{-2}} + e^T(0)\Pi(0)e(0)\Big)dt\end{aligned} \quad (3.26)$$

where $\gamma^2 = \overline{l}/(\lambda^{-2}\underline{l} - 2\kappa)$ is a positive real number if $\lambda^{-2}\underline{l} > 2\kappa$, and indicates the filter attenuation constant.

Clearly, (3.26) can be viewed as a modification of (2.13) in the sense that it incorporates the effects of model uncertainties and initial estimation errors, whereas (2.13) does not. This concludes the proof of Theorem 3.1. ■

It is noted that, γ is not only an index of disturbance attenuation level, but also an important parameter describing filter's estimation ability in the worst case. Decreasing γ will enhance the robustness of the filter. The SDRE-based H_∞ control offered in [12], is based on a game theoretic approach and exhibits robustness only against disturbances. However, the beauty of (3.26) is that it derives from a familiar Lyapunov-based approach and guarantees robustness against the system model uncertainty as well as the process and measurement noises.

We now endeavor to extend our results to a class of forced uncertain systems. Suppose the state equation (2.4) is controlled by the input $u(t) \in \mathbb{R}^l$ as follows

$$\dot{x}(t) = A(x)x + B(x)u + E_1(t)\Delta_1(t)N(x) + G(t)w_0(t) \qquad (3.27)$$

where $B(x) \in \mathbb{R}^{n\times l}$ is a known matrix function. Equation (3.27) together with (2.5) represents an uncertain form of the nonlinear control-affine system used in the SDRE control technique. We claim that, under certain conditions, the proposed filter will successfully work for the given forced system, as well. The following corollary evolves this fact.

Corollary 3.1: Let the control input $u(t)$ is norm-bounded, i.e., $\|u(t)\| \le \rho$ for some $\rho > 0$, and the control matrix $B(x)$ is also locally Lipschitz, i.e., $\|B(x) - B(\hat{x})\| \le k_B \|x - \hat{x}\|$ for $k_B > 0$ and $\|x - \hat{x}\| \le \varepsilon_B$. Then under the conditions of Theorem 3.1, applying (2.8)-(2.10), with an additive term of $B(\hat{x})u$ in (2.8), to the given system (3.27) and (2.5) achieves the same performance index as (3.26). The only discrepancy is that in this case, $\kappa = (k_A\sigma + k_B\rho)/\underline{p} + \overline{c}k_C\sigma/\underline{r}$ and $\varepsilon = \min(\varepsilon_A, \varepsilon_B, \varepsilon_C)$.

Proof: The proof is in analogy to that of Theorem 3.1, thus omitted. ■

III. ILLUSTRATIVE EXAMPLE

To exemplify the performance improvement of the proposed SDRE filter over the usual algebraic and differential SDRE filters, we consider a second-order nonlinear uncertain system expressed as

$$\dot{x}(t) = \begin{bmatrix} x_1^2 - 2x_1x_2 + (-1 + \delta_1(t))x_2 \\ x_1x_2 + x_2 \sin x_2 + (1 + \delta_2(t))x_1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w_0(t) \quad (4.1)$$

$$y(t) = x_1 + v_0(t) \qquad (4.2)$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and $\delta_1(t)$, $\delta_2(t)$ are unknown time varying functions satisfying the condition

$$\left\| \begin{bmatrix} \delta_1(t) \\ \delta_2(t) \end{bmatrix} \right\| \le 1 \qquad (4.3)$$

The disturbing noise signals $w_0(t)$ and $v_0(t)$ are drawn from two different distributions with unknown statistics. Precisely, $w_0(t)$ is a white Gaussian process noise while $v_0(t)$ is a uniformly distributed measurement noise.

Obviously, (4.1)-(4.2) takes the form of (2.1)-(2.2) and can be brought to the uncertain SDC form (2.4)-(2.5) by any suitable parameterization. Among several possible choices, let us set

$$A(x) = \begin{bmatrix} x_1 - 2x_2 & -1 \\ 1 & x_1 + \sin x_2 \end{bmatrix} \qquad (4.4)$$

$$C(x) = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad (4.5)$$

In addition, in this example there is no measurement uncertainty ($\Delta_2 \equiv 0$) while the state equation uncertainty is described by

$$\Delta_1(t) = \begin{bmatrix} 0 & \delta_1(t) \\ \delta_2(t) & 0 \end{bmatrix}, \quad N(x) = x \qquad (4.6)$$

with $E_1(t) = I_2$.

Note that (4.5) is a trivial choice, and one can choose other forms such as $C(x) = [1 + x_2 \quad x_1]$. The reason of our choices, (4.4) and (4.5), is mainly related to the compliance of inequalities (3.8), (3.10), and (3.11). First, it follows from (4.4) that for all $x, \hat{x} \in \mathbb{R}^2$,

$$A(x) - A(\hat{x}) = \begin{bmatrix} (x_1 - \hat{x}_1) - 2(x_2 - \hat{x}_2) & 0 \\ 0 & (x_1 - \hat{x}_1) + (\sin x_2 - \sin \hat{x}_2) \end{bmatrix}. \quad (4.7)$$

Since

$$\left|(x_1 - \hat{x}_1) - 2(x_2 - \hat{x}_2)\right| \le \sqrt{5}\,\|x - \hat{x}\|$$

and

$$\left|(x_1 - \hat{x}_1) + (\sin x_2 - \sin \hat{x}_2)\right| \le \sqrt{2}\,\|x - \hat{x}\|,$$

it can be deduced that (3.10) is globally valid with $k_A = \sqrt{5}$. Second, the selected output matrix of (4.5) fulfils (3.8) with $\overline{c} = 1$ and (3.11) with any positive real Lipschitz constant such as $k_C = 0.001$. Considering these facts together with the Lyapunov stability of (4.1), we conclude that assumptions 3.1 and 3.2 hold.

We implemented the robust SDRE filter according to (2.8)-(2.10) to obtain the desired performance for the system (4.1)-(4.2). The differential equations are solved numerically by the Runge-Kutta method, choosing the initial conditions $x(0) = \begin{bmatrix} -0.5 & 0.5 \end{bmatrix}^T$ for the system to be observed, $\hat{x}(0) = \begin{bmatrix} 0.5 & -0.5 \end{bmatrix}^T$ for the filter and $P(0) = 10I_2$ for the SDDRE (2.10). The design details are summarized below.

The filter output, $z(x)$ in (2.6), is assumed to be $x(t)$ itself. Therefore, in this case, $L(t)$ is the identity matrix and $\overline{l}, \underline{l}$ can be unity. We also choose the weighting matrices as $Q = I_2$ and $R = 0.1$. The appropriate values for λ and μ are obtained using a binary search algorithm similar to that of [21]. By this means, it was turned out that $\lambda = 0.5$ and $\mu = 0.004$ are sufficient for $P(t)$ in (2.10) to be always positive definite. Besides, this value of λ will satisfy (3.12). This fact can be easily verified by inserting the values of k_A, k_C, and $\overline{c}$, analytically determined above, along with $\underline{r} = 0.1$, $\sigma = 0.707$, and $\underline{p} = 10$ into (3.13) which yields $\kappa = 0.165$.

So far, all the sufficient conditions in Theorem 3.1 have been ensured and hence, it is expected to reach the modified H_∞ performance index obtained in (3.26). This is verified through the simulation results depicted in Fig. 1. The figure shows the true state of the system together with the estimated value obtained from the robust SDRE filter. It is clear that the filter performs as expected, and the estimated signals converge quickly to the corresponding actual ones in spite of the considered disturbances and modeling uncertainties.

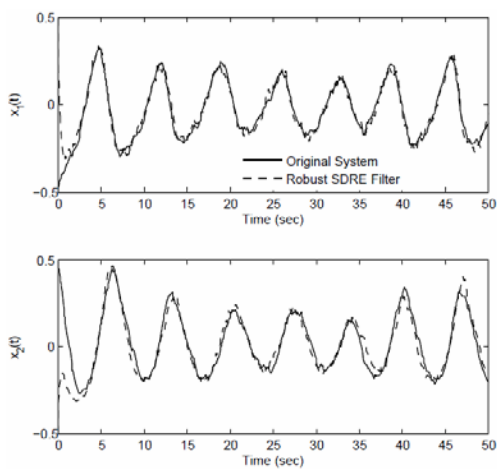

Fig. 1. The actual and the estimated states by the robust SDRE filter

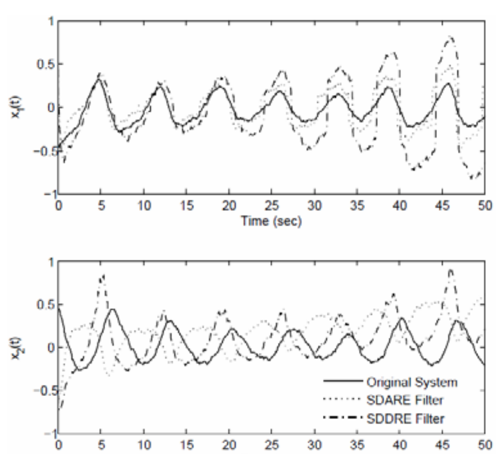

Fig. 2. The actual and the estimated states by the algebraic and differential SDRE filters

Note that according to (3.14), the given $\lambda = 0.5$ guarantees an attenuation level of $\gamma^2 = 0.27$. This means the energy gain from the disturbances to the estimation errors is bounded by 0.27.

For the sake of comparison, the standard algebraic and differential SDRE filters, namely SDARE filter and SDDRE filter, was also simulated with the same weighting matrices Q, R and the same initial conditions $\hat{x}(0)$. The results of this simulation are plotted in Fig. 2. It can be observed that in these two cases, the estimated signals do not track the true ones and exhibit divergence behaviors.

IV. CONCLUSION

To overcome the destructive effects of uncertain dynamics and unknown disturbance inputs on the performance of the usual SDRE filters, we developed a new robust H_∞ filter design which can be seen as a robustified differential SDRE filter. The proposed filter can be systematically applied to nonlinear continuous-time systems with an uncertain SDC form. We proved that under specific conditions the proposed filter guarantees the modified H_∞ performance criterion by choosing an appropriate Lyapunov function. This criterion is modified in the sense that it incorporates both the effects of disturbances and model uncertainties in the H_∞ norm minimization. Numerical simulations show the promising performance of the robust SDRE filter in comparison with the standard SDRE filters, which makes it a viable H_∞ filtering method.

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