

Stability Analysis of the Discrete-Time Difference SDRE State Estimator in a Noisy Environment

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Abstract—The state-dependent Riccati equation filter (SDREF) is a recent nonlinear estimation technique which has yielded a number of impressive results. However, the theoretical investigations of the filter have been carried out only in a deterministic environment. In this paper a discrete time difference SDRE-based observer for general nonlinear systems in a stochastic framework is considered, and its error behavior has been also analyzed. It is proved that, the estimation error remains bounded in mean square if the system to be observed satisfies certain conditions, and both the initial estimation error and the disturbing noise terms are small enough. Moreover, the results are verified through a simulation study of an example system.

I. INTRODUCTION

THE SDRE filter is one of the promising and rapidly emerging methodologies for designing nonlinear state estimators [1], [2] and has demonstrated its effectiveness in different applications through extensive numerical simulations [3]-[4]. In essence, the SDRE-based observer originates from a well-known suboptimal nonlinear regulator technique, called the SDRE control scheme. It uses parameterization to bring the nonlinear system into a linear-like structure with state-dependent coefficients (SDC) and obviates the need for Jacobian computations in the broadly acceptable extended Kalman filter (see, e.g., [5-6]). Moreover, due to the nonuniqueness of the SDC form in multivariable case [2], one can use this extra degree of freedom to address the loss of observability problem in traditional filtering techniques.

Indeed, the SDREF has the structure of the steady state Kalman filter and the Kalman gain is obtained by solving an algebraic Riccati equation (ARE) which can be computationally expensive for large scale systems. Furthermore, if loss of observability occurs during certain time-intervals, then the algebraic Riccati equation may not have a solution and the algebraic Riccati equation based SDREF cannot be used during these time-intervals [7].

A recently proposed SDRE-based observer is derived by removing the infinite time horizon assumption and using difference rather than algebraic Riccati equation [8]. This alternative addresses the issues of high computational load

and the potentially restrictive observability requirement in the algebraic form of the estimator.

Despite the superior practical usefulness of the SDREF, it has not been analyzed in a rigorous mathematical way for a long time. Banks *et al.* [1] have shown the local convergence of continuous-time algebraic based SDREF under certain Lipschitzian conditions and a particular splitting of the state dependent matrices. In [9], we have modified the differential SDRE based observer in order to obtain an exponential observer. Jaganath *et al.* [7] and Ewing [8] provide two distinct sufficient conditions sets for asymptotic stability of the discrete-time difference SDREF in a deterministic setting. However, in addition to these results a study of a more general nonlinear case in a stochastic framework would also be of some interest.

In this paper, motivated by the stability results for the usual Kaman-Bucy filter (see, e.g., [6], ch. 7) and the stochastic stability analysis for more general nonlinear estimation problems [10], [11], we analyze the error behavior of the discrete-time difference SDRE filter. The main contribution consists of the proof that, under certain conditions, the estimation error of the SDREF remains bounded. Similar idea has been used to analyze the error behavior of the extended Kalman filter (EKF) in [12] and also the Unscented Kalman filter (UKF) in [13].

The paper is organized as follows. In Section II we introduce the discrete-time difference SDREF and recall some auxiliary results from stochastic stability theory. Then, in Section III the error boundedness is proved if certain conditions are satisfied. Section IV contains an illustrative simulation example. Finally, some conclusions are drawn in Section V.

II. DIFFERENCE SDRE FILTER AND BACKGROUNDS

Consider the stochastic nonlinear discrete-time system affine in the input

$$x_{k+1} = f(x_k) + g(x_k)u_k + G_k w_k \quad (2.1)$$

$$y_k = h(x_k) + D_k v_k \quad (2.2)$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ the input and $y_k \in \mathbb{R}^l$ the output. Moreover, v_k and w_k are \mathbb{R}^q and \mathbb{R}^p valued uncorrelated zero-mean white noise processes, called process and measurement noise respectively, with identity covariance. D_k and G_k are time varying matrices of size $l \times q$ and $n \times p$. Assume that the nonlinear functions $f(x_k)$

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and $h(x_k)$ belong to C^1 and the nonlinear dynamics (2.1) and (2.2) can be put into the following state-dependent coefficient (SDC) form

$$x_{k+1} = A(x_k)x_k + B(x_k)u_k + G_k w_k \quad (2.3)$$

$$y_k = C(x_k)x_k + D_k v_k \quad (2.4)$$

in which $A(x_k)$, $B(x_k)$ and $C(x_k)$ are discrete $n \times n$, $n \times m$ and $l \times n$ matrix-valued functions, respectively.

For the dynamical system given by (2.3) and (2.4), let us introduce a state estimator as follows

$$\hat{x}_{k+1} = A(\hat{x}_k)\hat{x}_k + B(\hat{x}_k)u_k + L_k [y_k - C(\hat{x}_k)\hat{x}_k] \quad (2.5)$$

where the observer gain L_k is a matrix-valued stochastic process of size $n \times l$ and \hat{x}_k denotes the estimated state vector. We can now formulate the difference SDRE observer by taking the dual of the discrete-time SDRE control technique and removing the infinite time horizon assumption. Hence, consider the cost function

$$J(\tilde{x}_k, \tilde{u}_k) = \frac{1}{2} \sum_{k=0}^N (\tilde{x}_k^T Q_k \tilde{x}_k + \tilde{u}_k^T R_k \tilde{u}_k) \quad (2.6)$$

associated with the state dynamics of the dual system expressed as

$$\tilde{x}_{k+1} = A^T(\tilde{x}_k)\tilde{x}_k + C^T(\tilde{x}_k)\tilde{u}_k \quad (2.7)$$

where N is the finite time horizon, $Q_k \in \mathbb{R}^{n \times n}$ is a time varying symmetric positive definite matrix and $R_k \in \mathbb{R}^{l \times l}$ is a time varying positive definite matrix. By mimicking the theory of observers for linear systems and in order to minimize the forgoing cost function, we choose the observer gain, L_k , in equation (2.5) as follows

$$L_k = A(\hat{x}_k)P_k C^T(\hat{x}_k) [C(\hat{x}_k)P_k C^T(\hat{x}_k) + R_k]^{-1} \quad (2.8)$$

and P_k is updated using the difference state-dependent Riccati equation

$$P_{k+1} = A(\hat{x}_k)P_k A^T(\hat{x}_k) - A(\hat{x}_k)P_k C^T(\hat{x}_k) [C(\hat{x}_k)P_k C^T(\hat{x}_k) + R_k]^{-1} C(\hat{x}_k)P_k A^T(\hat{x}_k) + Q_k \quad (2.9)$$

Remarks:

2.1) If $A_1(x_k)$ and $A_2(x_k)$ are two distinct parameterizations of $f(x_k)$, they can be combined to yield a third parameterization by

$$A(x_k, \alpha) = \alpha A_1(x_k) + (1 - \alpha) A_2(x_k) \quad (2.10)$$

These additional degrees of freedom allow an infinite number of parameterization possibilities, which can be used to enhance the filter performance and avoid loss of observability [2].

2.2) Similar to the extended Kalman filter, there are two common formulations of the discrete-time difference SDRE filter, namely, the two-step recursive update

and the one-step recursive update, respectively. The one step recursive formulation is given by (2.5), (2.8) and (2.9). The two step recursive formulation in a deterministic setting is treated in [7] and also [8]. Note that, these two formulations may have different performances and transient behaviors, but the convergence properties are the same.

2.3) A usual choice for the matrices Q_k and R_k are the covariances for the corrupting noise terms in (2.1) and (2.2), i.e.,

$$Q_k = G_k G_k^T \quad (2.11)$$

$$R_k = D_k D_k^T \quad (2.12)$$

However, this is not the only possibility. Any other positive definite matrices can be chosen as well.

Let us define the estimation error by

$$e_k = x_k - \hat{x}_k \quad (2.13)$$

Subtracting (2.5) from (2.3) the error dynamics is obtained

$$e_{k+1} = A(x_k)x_k + B(x_k)u_k + G_k w_k - A(\hat{x}_k)\hat{x}_k - B(\hat{x}_k)u_k - L_k [C(x_k)x_k + D_k v_k - C(\hat{x}_k)\hat{x}_k] \quad (2.14)$$

adding and subtracting $A(\hat{x}_k)x_k$ to the whole equation and adding and subtracting $C(\hat{x}_k)x_k$ into the bracket lead to

$$e_{k+1} = A(\hat{x}_k)x_k - A(\hat{x}_k)\hat{x}_k + A(x_k)x_k - A(\hat{x}_k)x_k + [B(x_k) - B(\hat{x}_k)]u_k + G_k w_k - L_k [C(\hat{x}_k)x_k - C(\hat{x}_k)\hat{x}_k + C(x_k)x_k - C(\hat{x}_k)x_k + D_k v_k] \quad (2.15)$$

hence, the error dynamics can be rewritten as

$$e_{k+1} = [A(\hat{x}_k) - L_k C(\hat{x}_k)]e_k + \varphi_k + \chi_k \quad (2.16)$$

where φ_k and χ_k include the noise free and the noise compelled terms in (2.15), respectively

$$\varphi_k = d(x_k, \hat{x}_k, u_k) - L_k s(x_k, \hat{x}_k) \quad (2.17)$$

$$\chi_k = G_k w_k - L_k D_k v_k \quad (2.18)$$

and the nonlinear functions $d(x_k, \hat{x}_k, u_k)$, $s(x_k, \hat{x}_k)$ are given by

$$d(x_k, \hat{x}_k, u_k) = [A(x_k) - A(\hat{x}_k)]x_k + [B(x_k) - B(\hat{x}_k)]u_k \quad (2.19)$$

$$s(x_k, \hat{x}_k) = [C(x_k) - C(\hat{x}_k)]x_k \quad (2.20)$$

For the analysis of the error dynamics (2.16) we make use of the following two concepts for the boundedness of stochastic processes [10], [11].

Definition 2.1: The discrete stochastic process e_k is said to be exponentially bounded in mean square, if there exist real numbers $\nu, \eta > 0$ and $0 \leq \rho < 1$ such that

$$E\{\|e_k\|^2\} \leq \nu + \eta \|e_0\|^2 \mathcal{G}^k \quad (2.21)$$

holds for every $k \geq 0$.

Definition 2.2: The stochastic process is said to be bounded with probability one, if

$$\sup_{k \geq 0} \|e_k\| < \infty \quad (2.22)$$

holds with probability one.

III. BOUNDEDNESS OF THE ESTIMATION ERROR

The stability analysis of the difference SDRE filter is based on the following lemma which is a standard result about the boundedness of stochastic processes.

Lemma 3.1: Assume that e_k is a stochastic process and there is a stochastic function $V_k(e_k)$ as well as real numbers $\bar{\nu}, \underline{\nu}, \mu > 0$ and $0 < \lambda \leq 1$ such that

$$\underline{\nu} \|e_k\|^2 \leq V_k(e_k) \leq \bar{\nu} \|e_k\|^2 \quad (3.1)$$

and

$$E\{V_{k+1}(e_{k+1}) | e_k\} - V_k(e_k) \leq \mu - \lambda V_k(e_k) \quad (3.2)$$

are fulfilled for every solution of (2.16). Then the stochastic e_k is exponentially bounded in mean square, that is

$$E\{\|e_k\|^2\} \leq \frac{\bar{\nu}}{\underline{\nu}} E\{\|e_0\|^2\} (1-\lambda)^k + \frac{\mu}{\underline{\nu}} \sum_{i=1}^{k-1} (1-\lambda)^i \quad (3.3)$$

for every $k \geq 0$. Moreover, the stochastic process is bounded with probability one.

Proof: This lemma follows from [10], Theorem 2 (See also [12], [13] and the references cited therein) \square

Note that this lemma contains Lyapunov-like conditions for stochastic stability. Intuitively, if condition (3.2) is fulfilled, the energy e_k will not increase arbitrarily.

Remark 3.1: In the sequel, the state-dependent matrices appearing in the filter formulation are replaced by the following time varying matrices

$$A_k \triangleq A(\hat{x}_k) \quad B_k \triangleq B(\hat{x}_k) \quad C_k \triangleq C(\hat{x}_k) \quad (3.4)$$

Note that, this is done just for notational convenience and we don't ignore the fact that, these matrices are obtained using the state \hat{x}_k .

In order to set up the error analysis, we first make the following assumptions.

Assumption 3.1: There are positive real numbers $\bar{c}, \bar{a} > 0$ such that, for all $k \geq 0$, the matrices A_k and C_k are bounded by

$$\|A_k\| \leq \bar{a} \quad (3.5)$$

$$\|C_k\| \leq \bar{c} \quad (3.6)$$

and furthermore, A_k is nonsingular for every $k \geq 0$.

Assumption 3.2: For any solution \hat{x}_k of the observer difference equation (2.5), the solution P_k of the difference state-dependent Riccati equation (2.9) is bounded for every $k \geq 0$ via

$$\underline{p}I \leq P_k \leq \bar{p}I \quad (3.7)$$

for some positive real numbers $\underline{p}, \bar{p} > 0$.

Assumption 3.3: There exist $\sigma, \rho > 0$ such that, for all $k \geq 0$

$$\|x_k\| \leq \sigma, \|u_k\| \leq \rho \quad (3.8)$$

Assumption 3.4: The SDC parameterization is chosen such that, for all $x_k, \hat{x}_k \in \mathbb{R}^n$ and every $k \geq 0$ the following inequalities are satisfied

$$\|A(x_k) - A(\hat{x}_k)\| \leq k_A \|x_k - \hat{x}_k\|^2 \quad (3.9)$$

$$\|B(x_k) - B(\hat{x}_k)\| \leq k_B \|x_k - \hat{x}_k\|^2 \quad (3.10)$$

$$\|C(x_k) - C(\hat{x}_k)\| \leq k_C \|x_k - \hat{x}_k\|^2 \quad (3.11)$$

with $\|x_k - \hat{x}_k\| \leq \varepsilon_A$, $\|x_k - \hat{x}_k\| \leq \varepsilon_B$ and $\|x_k - \hat{x}_k\| \leq \varepsilon_C$, respectively.

Theorem 3.1: Consider a stochastic nonlinear system given by equations (2.3) and (2.4) and the difference SDRE filter defined by equations (2.5)-(2.9) with positive definite matrices Q_k and R_k . Let Assumptions 3.1-3.4 hold. Then the estimation error e_k given by (2.16) is exponentially bounded in mean square and bounded with probability one, provided that the initial estimation error satisfies

$$\|e_0\| \leq \varepsilon \quad (3.12)$$

and the covariance matrices of the noise terms are bounded via

$$G_k G_k^T \leq \delta I \quad (3.13)$$

$$D_k D_k^T \leq \delta I \quad (3.14)$$

for some $\varepsilon, \delta > 0$.

Remarks:

3.1) The inequalities (3.5)-(3.8) should be understood in the sense that they can be verified during the state estimation process.

3.2) For many applications the state variables, which often represent physical quantities, are bounded. Boundedness of the control input seems also a trivial hypothesis. Thus, Assumption 3.3 is satisfied easily. Moreover, if A_k and C_k fulfill (3.5) and (3.6), respectively, for every physical reasonable value of the state vector \hat{x}_k , we may suppose without loss of generality that, Assumption 3.1 is also satisfied.

3.3) It can be shown that Assumption 3.2 holds if the nonlinear system (2.1)-(2.2) satisfies certain observability and detectability properties [7], [8].

The related discussion for the continuous-time counterpart of the filter can be seen in [9].

3.4) The inequalities (3.9)-(3.11) resemble the Lipschitzian-like conditions imposed in [1] and may be difficult to fulfill them. Consequently, Assumption 3.4 is the key condition of Theorem 3.1. However, additional degrees of freedom provided by nonuniqueness of the SDC parameterization (see remark 2.1) can be exploited to satisfy these inequalities.

3.5) Theorem 3.1 provides sufficient conditions to guarantee the exponential boundedness of the estimation error. Furthermore, since the bounds obtained the norm operators these conditions may be conservative as well. However, the proof of this theorem contains a constructive way to quantify the error bounds according to (3.3) with $\|e_0\| \leq \varepsilon$, $\underline{v} = 1/\bar{p}$, $\bar{v} = 1/\underline{p}$ and $\mu = \kappa_\chi \delta$ (See Appendix A.)

To prove Theorem 3.1, we state the following preparatory lemmas.

Lemma 3.2: Under the conditions of Theorem 3.1, there are positive real numbers $\varepsilon', \kappa_\varphi > 0$ such that $\Pi_k = P_k^{-1}$ satisfies the inequality

$$\varphi_k^T \Pi_{k+1} [2(A_k - L_k C_k) e_k + \varphi_k] \leq \kappa_\varphi \|e_k\|^3 \quad (3.15)$$

for $\|e_k\| \leq \varepsilon'$ with L_k, φ_k given by (2.8), (2.17).

Proof: Denote the smallest eigenvalue of the positive-definite time varying matrices Q_k and R_k by \underline{q} and \underline{r} , respectively, then we have

$$\underline{q}I \leq Q_k \quad (3.16)$$

$$\underline{r}I \leq R_k \quad (3.17)$$

From (2.8), (3.5)-(3.7), (3.16), (3.17) and considering $C_k P_k C_k^T > 0$ we have

$$\|L_k\| \leq \overline{apc} \frac{1}{\underline{r}} \quad (3.18)$$

Inserting into (2.17) yields

$$\|\varphi_k\| \leq \|d(x_k, \hat{x}_k, u_k)\| + \overline{apc} \frac{1}{\underline{r}} \|s(x_k, \hat{x}_k)\| \quad (3.19)$$

Considering (2.19) and (2.20) then According to the inequalities (3.9)-(3.11) and using (3.8) we get

$$\|d(x_k, \hat{x}_k, u_k)\| \leq \left\| [A(x_k) - A(\hat{x}_k)] x_k \right\| + \left\| [B(x_k) - B(\hat{x}_k)] u_k \right\| \leq (k_A \sigma + k_B \rho) \|x_k - \hat{x}_k\|^2 \quad (3.20)$$

$$\|s(x_k, \hat{x}_k)\| = \left\| [C(x_k) - C(\hat{x}_k)] x_k \right\| \leq k_C \sigma \|x_k - \hat{x}_k\|^2 \quad (3.21)$$

Choosing $\varepsilon' = \min(\varepsilon_A, \varepsilon_B, \varepsilon_C)$ and using (3.20), (3.21) we rewrite (3.19) as

$$\|\varphi_k\| \leq (k_A \sigma + k_B \rho) \|e_k\|^2 + \overline{apc} \frac{1}{\underline{r}} k_C \sigma \|e_k\|^2 \quad (3.22)$$

for $\|e_k\| \leq \varepsilon'$, i.e.,

$$\|\varphi_k\| \leq \kappa' \|e_k\|^2 \quad (3.23)$$

with

$$\kappa' = (k_A \sigma + k_B \rho) + \overline{apc} \frac{1}{\underline{r}} k_C \sigma \quad (3.24)$$

The definition of the estimation error given by (2.13) has been used in (3.22). From (3.23), (3.5)-(3.7) and (3.17) we get with $\Pi_k = P_k^{-1}$ for $\|e_k\| \leq \varepsilon'$

$$\begin{aligned} & \varphi_k^T \Pi_{k+1} [2(A_k - L_k C_k) e_k + \varphi_k] \\ & \leq \kappa' \|e_k\|^2 \frac{1}{\underline{p}} \left[2 \left(\bar{a} + \overline{apc} \frac{1}{\underline{r}} \bar{c} \right) \|e_k\| + \kappa' \varepsilon' \|e_k\| \right] \end{aligned} \quad (3.25)$$

i.e., (3.15) is fulfilled with

$$\kappa_\varphi = \kappa' \frac{1}{\underline{p}} \left[2 \left(\bar{a} + \overline{apc} \frac{1}{\underline{r}} \bar{c} \right) + \kappa' \varepsilon' \right] \quad (3.26)$$

□

Lemma 3.3: Let the assumptions of Theorem 3.1 hold. Then, it can be shown that the following two inequalities are satisfied

$$(A_k - L_k C_k)^T \Pi_{k+1} (A_k - L_k C_k) \leq (1 - \lambda) \Pi_k \quad (3.27)$$

$$E \left\{ \chi_k^T \Pi_{k+1} \chi_k \right\} \leq \kappa_\chi \delta \quad (3.28)$$

where L_k, φ_k and χ_k are given by (2.8), (2.17) and (2.18), respectively. Note that, $0 < \lambda < 1$ and $\kappa_\chi > 0$ are positive real numbers which can be obtained in terms of various bounds in Theorem 3.1 and Assumptions 3.1-3.4, through some straight forward calculations (see [12], Lemmas 3.1 and 3.3).

Proof of Theorem 3.1: See Appendix A. □

Remarks 3.6: It can be shown that, inequalities (3.9)-(3.11) in Assumption 3.4 can be reduced to common Lipschitzian conditions with unit exponent, while the estimation error still remains bounded, provided that

$$\bar{p} + 1 / \left(1 + \underline{q} / \bar{p} (\bar{a} + \overline{apc}^2 / \underline{r})^2 \right) < 1 \quad (3.29)$$

The proof of Theorem 3.1, (See Appendix A.), can be modified easily for this case.

IV. NUMERICAL SIMULATIONS

In this section, to illustrate the significance of the conditions in the preceding section, we apply the difference SDRF to an example system and verify the error behavior by numerical simulations. For this purpose consider an

unforced, $u_k \equiv 0$, nonlinear stochastic example system given by (2.1), (2.2) with

$$f(x_k) = \begin{bmatrix} x_{1,k} + \tau x_{2,k} \\ (1-\tau)x_{2,k} + \tau(x_{1,k}^2 x_{2,k} + x_{2,k}^3 - x_{1,k}) \end{bmatrix} \quad (4.1)$$

$$h(x_k) = \begin{bmatrix} x_{1,k} + x_{2,k} \\ x_{1,k} x_{2,k} \end{bmatrix} \quad (4.2)$$

where $\tau = 10^{-2}$ is the sampling time. Among different possible choices we parameterize $f(x_k)$ as follows

$$f(x_k) = A(x_k)x_k = \begin{bmatrix} 1 & \tau \\ -\tau & 1 + \tau(x_{1,k}^2 + x_{2,k}^2 - 1) \end{bmatrix} x_k \quad (4.3)$$

In EKF formulation (see, e.g., [5]) the nonlinear observation function $h(x_k)$ has to be linearized about \hat{x}_k which leads to

$$H(\hat{x}_k) = \frac{\partial h(\hat{x})}{\partial \hat{x}} = \begin{bmatrix} 1 & 1 \\ \hat{x}_{2,k} & \hat{x}_{1,k} \end{bmatrix} \quad (4.4)$$

Note that $H(\hat{x})$ loses rank whenever $\hat{x}_{1,k} = \hat{x}_{2,k}$. It can be verified that the system appears unobservable to the EKF algorithm in this situation and there is nothing that can be done about it (see [2], [8] for similar example). However for the SDREF, there exist two distinct SDC parameterizations:

$$C_1(x_k) = \begin{bmatrix} 1 & 1 \\ 0 & x_{1,k} \end{bmatrix} \quad C_2(x_k) = \begin{bmatrix} 1 & 1 \\ x_{2,k} & 0 \end{bmatrix} \quad (4.5)$$

which can be combined to form the parameterized state-dependent coefficient measurement matrix as

$$C(x_k, \alpha) = \begin{bmatrix} 1 & 1 \\ \alpha x_{2,k} & (1-\alpha)x_{1,k} \end{bmatrix} \quad (4.6)$$

In this form, the value of α can be chosen such that loss of observability is avoided. Besides, it can be easily checked that Assumption 3.4 holds with (4.3) and (4.6).

For the numerical simulations, one case with bounded estimation error and two cases with divergent estimation error are considered. For all cases, the difference SDREF is implemented using the design parameters in Table I and the state-dependent matrices given by (4.3) and (4.6). Furthermore, the actual initial state is $x_0 = [0.8 \ 0.4]^T$ and the process noise covariance is set to $G_k = \sqrt{10^{-3}} I_2$. The remaining matrix D_k , as well as the initial value \hat{x}_0 , are chosen particularly for each of the three cases.

TABLE I
DIFFERENCE SDREF PARAMETERS

Design Parameters	Q_k	R_k	P_0	α
Value	diag(0.05,0.1)	100I ₂	I ₂	0.8

The simulation results are depicted in Figs. 1-4. Conditions (3.5)-(3.8) are verified by numerical simulations yielding $\bar{a} = 1.0001$, $\bar{c} = 1.4469$, $\bar{p} = 0.6806$, $\bar{p} = 1.0051$, and $\sigma = 0.9746$, respectively. As it can be seen in Figs. 1 and 2, for small initial error and small noise (cf. eqns. 3.12-3.14) the estimation error remains bounded. However, because of the high nonlinearities of the example system considered, if the initial estimation error or the disturbing noise is large, i.e., eqns. (3.12)-(3.14) are violated, then the estimation error is no longer bounded (see Figs. 3 and 4).

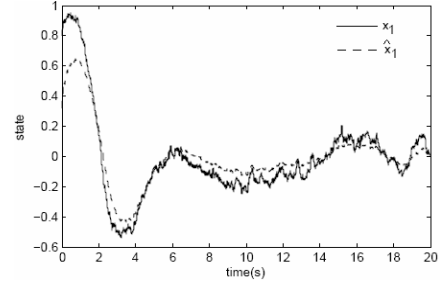


Fig. 1. The actual and the estimated value of the first state component for small initial error and small noise, $\hat{x}_0 = [0.3 \ 0.3]^T$ and $D_k = I_2$.

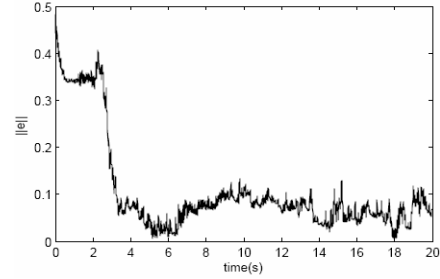


Fig. 2. The norm of the estimation error for small initial error and small noise

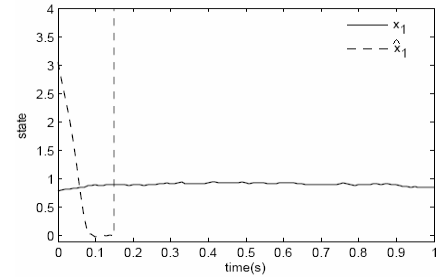


Fig. 3. The actual and the estimated value of the first state component for large initial error, $\hat{x}_0 = [3 \ 3]^T$.

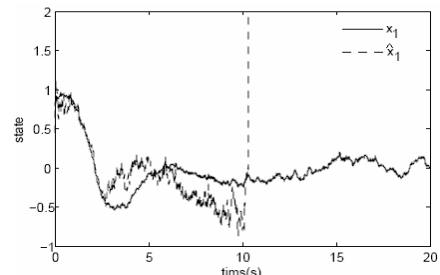


Fig. 4. The actual and the estimated value of the first state component for large measurement noise, $D_k = \sqrt{200} I_2$.

V. CONCLUSION

In this paper, a discrete-time difference SDRE observer for general nonlinear state estimation problems in a noisy environment is considered and its error behavior has been analyzed. It has been shown that under certain conditions, the estimation error is bounded in mean square and bounded with probability one. This fact is embodied in Theorem 3.1 in Section III. These conditions include the requirements that the SDC parameterization satisfy a Lipschitzian condition, the solution of the state-dependent Riccati difference equation remains positive definite and bounded, and furthermore the initial estimation error as well as the corrupting noise terms are small enough. The numerical simulations in Section IV indicate that the estimation error may diverge if either the initial error or the noise terms are large.

APPENDIX A. PROOF OF THEOREM 3.1

Choose

$$V_k(e_k) = e_k^T \Pi_k e_k \quad (\text{A.1})$$

with $\Pi_k = P_k^{-1}$, which exists with probability, since P_k is positive definite according to equation (3.7). From (3.7) it follows that

$$\frac{1}{\bar{p}} \leq \Pi_k \leq \frac{1}{\underline{p}} \quad (\text{A.2})$$

and with (A.1) we have

$$\frac{1}{\bar{p}} \|e_k\|^2 \leq V_k(e_k) \leq \frac{1}{\underline{p}} \|e_k\|^2 \quad (\text{A.3})$$

i.e., (3.1) with $\underline{v} = 1/\bar{p}$ and $\bar{v} = 1/\underline{p}$. To satisfy the requirements for an application of Lemma 3.1, we need an upper bound on $E\{V_{k+1}(e_{k+1}) | e_k\}$ as in (3.3). From (2.16) we have

$$V_{k+1}(e_{k+1}) = \left[e_k^T (A_k - L_k C_k)^T + \varphi_k^T + \chi_k^T \right] \Pi_{k+1} \left[(A_k - L_k C_k) e_k + \varphi_k + \chi_k \right] \quad (\text{A.4})$$

which leads to

$$\begin{aligned} V_{k+1}(e_{k+1}) &= e_k^T (A_k - L_k C_k)^T \Pi_{k+1} (A_k - L_k C_k) e_k \\ &+ \varphi_k^T \Pi_{k+1} [2(A_k - L_k C_k) e_k + \varphi_k] \\ &+ 2\chi_k^T \Pi_{k+1} [(A_k - L_k C_k) e_k + \varphi_k] + \chi_k^T \Pi_{k+1} \chi_k \end{aligned} \quad (\text{A.5})$$

We have used the symmetry property of P_k in (A.5). By applying (3.27) to (A.5), we obtain with (A.1)

$$\begin{aligned} V_{k+1}(e_{k+1}) &\leq (1-\lambda)V_k(e_k) + \varphi_k^T \Pi_{k+1} [2(A_k - L_k C_k) e_k \\ &+ \varphi_k] + 2\chi_k^T \Pi_{k+1} [(A_k - L_k C_k) e_k + \varphi_k] + \chi_k^T \Pi_{k+1} \chi_k \end{aligned} \quad (\text{A.6})$$

Taking the conditional expectation $E\{V_{k+1}(e_{k+1}) | e_k\}$ and considering the white noise property it can be seen that the term $E\{\chi_k^T \Pi_{k+1} [(A_k - L_k C_k) e_k + \varphi_k] e_k\}$ vanishes since

neither $\Pi_{k+1} = P_{k+1}^{-1}$ nor $A_k, C_k, L_k, \varphi_k, e_k$ depend on v_k or w_k . The remaining terms are estimated via Lemma 3.2 and inequality (3.28) and we get

$$E\{V_{k+1}(e_{k+1}) | e_k\} \leq (1-\lambda)V_k(e_k) + \kappa_\varphi \|e_k\|^3 + \kappa_\chi \delta \quad (\text{A.7})$$

for $\|e_k\| \leq \varepsilon'$. Defining

$$\varepsilon = \min(\varepsilon', \frac{\lambda}{2\bar{p}\kappa_\varphi}) \quad (\text{A.8})$$

we obtain with (A.1), (A.2) for $\|e_k\| \leq \varepsilon$

$$\kappa_\varphi \|e_k\| \|e_k\|^2 \leq \frac{\lambda}{2\bar{p}} \|e_k\|^2 \leq \frac{\lambda}{2} V_k(e_k) \quad (\text{A.9})$$

Inserting into (A.7) yields

$$E\{V_{k+1}(e_{k+1}) | e_k\} - V_k(e_k) \leq -\frac{\lambda}{2} V_k(e_k) + \kappa_\chi \delta \quad (\text{A.10})$$

for $\|e_k\| \leq \varepsilon$. Therefore, with (A.3) and (A.10) one can apply Lemma 3.1 with $\|e_0\| \leq \varepsilon$, $\underline{v} = 1/\bar{p}$, $\bar{v} = 1/\underline{p}$, $\mu = \kappa_\chi \delta$, and establish mean square exponential boundedness of the estimation error under the conditions of equations (3.12)-(3.14). \square

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