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Iterative-Analytic Redundancy Resolution Scheme for A Cable-Driven Redundant Parallel Manipulator

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***Abstract*— In this paper, redundancy resolution of a cable-driven parallel manipulator is solved by an iterative-analytic scheme. The method can be applied to all kind of redundant manipulators either parallel or serial with constraint caused through their dynamics. However, for sake of simulation the proposed method is implemented on a cable-driven redundant parallel manipulator (CDRPM). The redundancy resolution problem is formulated as a convex optimization with equality and non-equality constraints caused by manipulator structure and cables dynamics. Karush-Kuhn-Tucker theorem is used to analyze the optimization problem and to find an analytic solution. Subsequently, a tractable and iterative search algorithm is proposed to solve the redundancy resolution of such redundant mechanisms. Furthermore, it is shown through the simulation that, the elapsed time required to implement the analytical redundancy resolution scheme in a closed-loop structure is considerably less than that of other numerical optimization methods.**

I. Introduction

WHILE redundancy is a desirable feature in robot manipulators, the presence of redundant actuators will considerably complicate the manipulator control. Redundancy resolution plays a crucial role in manipulators design, and in the control of robotic manipulators, and therefore, the redundancy resolution techniques have been extensively developed during past three decades. Despite this long history, previous investigation are often focused on the Jacobian pseudo-inverse approach proposed originally by Whitny [1] in 1969 and improved subsequently by Liegise [2]. This approach resolves the redundancy at the velocity level by optimization routines applied on some objective functions. Due to rigidity of serial manipulators structure, this method works well to produce minimum norm forces in the joint space, and therefore, the required energy from the actuators are minimized. By using Jacobian pseudo-inverse approach, Suh and Hollerbach [3] have suggested methods for minimizations of instantaneous joint torques. Khatib [4] has proposed a scheme to reduce joint torques through inertia-weighted Jacobian pseudo-inverse. Dubey and Luh [5] and Chiu [6] used the pseudo-inverse approach to optimize the manipulator mechanical advantage and velocity-ratio using the force and velocity manipulability ellipsoids. Maciejewski and Klein [7], described a method for obstacle avoidance based on pseudo-inverse control. Seraji, proposed a configuration control approach for redundancy resolution of serial manipulators [8].

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In other researches, optimization techniques are used to solve the redundancy resolution problem with an integral type performance index [9], and others have used Pontryagin's maximum principle for solution [10]. Some of these methods would automatically generate trajectories that avoid kinematic singularities [11]. Another approach is maximizing some function of the joint angles, such as the *manipulability measure* [12]. Similar to the open-chain serial manipulators, the redundancy resolution of parallel manipulators presents an inherent complexity due to their dynamics constraints, particularly when parallel manipulator is cable-driven and the cables dynamics restriction are added to the manipulator behavior [13].

It is important to note that, in implementation of all the above optimal techniques, numerical and iterative approaches are the common way to perform the optimizations [14]. On the other hand, in order to use such techniques in closed-loop control algorithm, it is required to solve the problem in real time, and therefore, the optimization routine must converge to a solution in a fixed and small period of time. However, this is in direct contrast to generic numerical algorithms, which take a variable time step time and exit only when a certain precision has been achieved. The main benefit of having an iterative-analytic solution to the redundancy resolution is to guarantee that the amount of time required for the overall solution remains within an acceptable and small period of time that can be used in real time implementation of closed-loop system.

In this paper the analytic solution of the redundancy resolution problem and its implementation on a cable-driven redundant manipulator is studied in detail. This is formulated into an optimization problem with equality and non-equality constraints. Nonlinear programing techniques and particularly Karush-Kuhn-Tucker theorem is used to analyze this optimization problem and to generate an analytic solution. Subsequently, a tractable and iterative algorithm is proposed to effectively generate such solutions. It is shown through simulations that, the elapsed time required to implement the iterative-analytic redundancy resolution scheme is considerably less than that of other numerical optimization methods.

II. Redundancy Resolution

For a cable-driven parallel manipulator the Jacobian matrix, $\boldsymbol{J_t}$, is defined as the projection matrix of the vector of actuator velocities, $\dot{\boldsymbol{q}}$, to the vector of manipulator moving platform velocity, $\dot{\boldsymbol{x}}$ as [15]:

$$\dot{\boldsymbol{q}}_{\boldsymbol{n}\times\boldsymbol{1}} = (\boldsymbol{J_t})_{\boldsymbol{n}\times\boldsymbol{m}}.\dot{\boldsymbol{x}}_{\boldsymbol{m}\times\boldsymbol{1}} \tag{1}$$

In which, n is the number of actuators, and m is the number of degrees of freedom of the robot. It is well known that Jacobian matrix not only reveals the relation between the velocity variables $\dot{\boldsymbol{q}}$ and $\dot{\boldsymbol{x}}$, but also constructs the transformation needed to find the actuator forces $\boldsymbol{\tau}_{\boldsymbol{n}\times\boldsymbol{1}}$ from the forces acting on the moving platform $\boldsymbol{F}_{\boldsymbol{m}\times\boldsymbol{1}}$.

$$\boldsymbol{F}_{\boldsymbol{m}\times\boldsymbol{1}} = (\boldsymbol{J}_{\boldsymbol{t}})^{\boldsymbol{T}}_{\boldsymbol{m}\times\boldsymbol{n}} \cdot \boldsymbol{\tau}_{\boldsymbol{n}\times\boldsymbol{1}} \tag{2}$$

Due to the redundancy in actuators, $\boldsymbol{J}_{\boldsymbol{t}}$ in Eq. (2) is a non-square matrix with $m < n$. If the manipulator has no redundancy in actuators, the Jacobian matrix $\boldsymbol{J}_{\boldsymbol{t}}$, is squared and the actuator forces can be uniquely determined by $\boldsymbol{\tau} = \boldsymbol{J}_{\boldsymbol{t}}^{-\boldsymbol{T}} \cdot \boldsymbol{F}_{\boldsymbol{n}\times\boldsymbol{1}}$, provided that $\boldsymbol{J}_{\boldsymbol{t}}$ is nonsingular. For redundant manipulators, however, there are infinitely many solution for $\boldsymbol{\tau}$ to be projected into $\boldsymbol{F}_{\boldsymbol{n}\times\boldsymbol{1}}$. The simplest solution would be a minimum norm solution, which is found from the pseudo–inverse of $\boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}}$. The Penrose-Moore pseudo-inverse of Jacobian matrix $\boldsymbol{J}$ that is denoted by $\boldsymbol{J}^{\dagger}$ is calculated as following.

$$\boldsymbol{J}^{\dagger} \triangleq \boldsymbol{J}^{T} \cdot (\boldsymbol{J} \cdot \boldsymbol{J}^{T})^{-1} \quad for \quad m < n \ ; \quad \boldsymbol{J} \in R^{m\times n} \tag{3}$$

By this means, the actuator forces can be simply obtained through:

$$\boldsymbol{\tau}_{\boldsymbol{n}\times\boldsymbol{1}} = (\boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}})^{\dagger} \cdot \boldsymbol{F}_{\boldsymbol{m}\times\boldsymbol{1}} \quad \boldsymbol{for} \ \boldsymbol{m} < n \tag{4}$$

This simple solution of the redundancy resolution problem chooses the minimum norm solution for the actuators among many solutions that satisfy $\boldsymbol{F} = (\boldsymbol{J}_{\boldsymbol{t}})^{\boldsymbol{T}} \cdot \boldsymbol{\tau}$. However, this solution does not ensure positive actuator forces in the cables. Therefore, among the optimal solutions of the redundancy resolution which satisfy the main equality constraint of the projection Eq.(2). The ones that satisfy the following non-equality constraint are of interest for the cable-driven manipulators.

$$\boldsymbol{\tau}_{\boldsymbol{n}\times\boldsymbol{1}} \geq \boldsymbol{F}_{\boldsymbol{min}} \tag{5}$$

Choosing $\boldsymbol{F}_{\boldsymbol{min}}$ to be a non-negative constant $\boldsymbol{F}_{\boldsymbol{min}} \geq \boldsymbol{0}$, this inequality constraint ensures that the cables are all in tension. In this view, redundancy resolution problem for cable-driven manipulators can be reformulated into an *optimization* problem with *equality* and *non-equality Constraints*. Nonlinear programming methods are used to solve such problem, and *Karush-Kuhn-Tucker Theorem* has served as the basis of those solutions.

A. *Karush-Kuhn-Tucker Theorem*

Implementation of nonlinear programming theorems, especially Karush-Kuhn-Tucker theorem is directly used in redundancy resolution techniques developed for serial redundant manipulators [16]. This optimization problem must be satisfy the projection map as of the Eq.(2) equivalent constraint and in addition to provide only the positive tension forces in the cables as of Eq.(5) non-equality constraint. Therefore it is possible to release the redundancy resolution problem as the problem of computing the minimum norm actuator forces with the following equality and non-equality constraints:

$$min\|\boldsymbol{\tau}_{\boldsymbol{n}\times\boldsymbol{1}}\|_2$$

Under the constraints:

$$\begin{gathered}\boldsymbol{F}_{\boldsymbol{m}\times\boldsymbol{1}} = (\boldsymbol{J}_{\boldsymbol{t}})^{\boldsymbol{T}}_{\boldsymbol{m}\times\boldsymbol{n}} \cdot \boldsymbol{\tau}_{\boldsymbol{n}\times\boldsymbol{1}} \\ \boldsymbol{\tau}_{\boldsymbol{n}\times\boldsymbol{1}} \geq \boldsymbol{F}_{\boldsymbol{min}}\end{gathered} \tag{6}$$

If there is a solution for this optimization problem, it can be written in the following general form:

$$\boldsymbol{\tau}_{\boldsymbol{n}\times\boldsymbol{1}} = \boldsymbol{F}_{\boldsymbol{0}} + \boldsymbol{A}\boldsymbol{y} \qquad ; \boldsymbol{y} \in R^{b} \tag{7}$$

in which, $\boldsymbol{F}_{\boldsymbol{0}}$ is the force that is projected by the pseudo-inverse of the Jacobian matrix as define by the following equation and has minimum norm property but not certainly positive:

$$\boldsymbol{F}_{\boldsymbol{0}} = (\boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}})^{\dagger} \cdot \boldsymbol{F}_{\boldsymbol{m}\times\boldsymbol{1}} \tag{8}$$

In this equation $(\boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}})^{\dagger}$ is the Pseudo-Inverse of Jacobian matrix $\boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}}$ defined by Eq. (2). Furthermore, *b* is the dimension of the null space of $\boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}}$, and $\boldsymbol{A}$ is the augmented matrix build with orthonormal column vectors of the null space of $\boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}}$.

$$\boldsymbol{A} = orthonormal \ \{nullspace \ of \ (\boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}})\} \tag{9}$$

It can be inferred that matrix $\boldsymbol{A}$ may be generated by collecting the linearly independent column vectors of real square matrix $\boldsymbol{M} = \boldsymbol{I}_{\boldsymbol{m}\times\boldsymbol{m}} - (\boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}})^{\dagger} \cdot \boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}}$ and then make it onrthonormal.

$$\boldsymbol{A} = orthonormal \ \{\boldsymbol{I}_{\boldsymbol{m}\times\boldsymbol{m}} - (\boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}})^{\dagger} \cdot \boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}}\} \tag{10}$$

This can be done using regular QR decomposition of real square matrix $\boldsymbol{M} = \boldsymbol{I}_{\boldsymbol{m}\times\boldsymbol{m}} - (\boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}})^{\dagger} \cdot \boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}}$. QR decomposition of a matrix, is a decomposition of the matrix into an orthogonal and an upper triangular matrix as

$$\boldsymbol{M} = \boldsymbol{Q} \cdot \boldsymbol{R} \tag{11}$$

in which, $\boldsymbol{Q}$ is an orthogonal matrix, i.e. $\boldsymbol{Q}^{\boldsymbol{T}} \cdot \boldsymbol{Q} = \boldsymbol{I}$, and $\boldsymbol{R}$ is an upper triangular matrix [17]. If $\boldsymbol{M}$ is nonsingular, then this factorization is unique. There are several tractable numerical methods in order to efficiently compute the QR decomposition. Therefore, in our formulation matrix $\boldsymbol{A}$ can be tractably computed by QR decomposition of matrix $\boldsymbol{M}$ as:

$$\boldsymbol{A} = QR \ decomposition \ of \ \{\boldsymbol{I}_{\boldsymbol{m}\times\boldsymbol{m}} - (\boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}})^{\dagger} \cdot \boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}}\} \tag{12}$$

Returning to the general solution of the redundancy resolution problem the first term of Eq.(7), namely $\boldsymbol{F}_{\boldsymbol{0}}$, can be viewed as an element of orthogonal complement of the null space of $\boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}}$ (range of $\boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}}$) and the second term $\boldsymbol{A}\boldsymbol{y}$ as an element of the null space of $\boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}}$. Now let us rewrite the optimization objective Eq.(6) as:

$$\begin{aligned}\|\boldsymbol{\tau}_{\boldsymbol{n}\times\boldsymbol{1}}\|_2 &= \left(\boldsymbol{\tau}_{\boldsymbol{n}\times\boldsymbol{1}}\right)^{T} \cdot (\boldsymbol{\tau}_{\boldsymbol{n}\times\boldsymbol{1}}) \\ &= (\boldsymbol{F}_{\boldsymbol{0}} + \boldsymbol{A}\boldsymbol{y})^{T} \cdot (\boldsymbol{F}_{\boldsymbol{0}} + \boldsymbol{A}\boldsymbol{y})\end{aligned} \tag{13}$$

Therefore, Eq.(6) can also be represented by:

$$\min_{y} \ f(\boldsymbol{y}) = (\boldsymbol{F}_{\boldsymbol{0}} + \boldsymbol{A}\boldsymbol{y})^{T} \cdot (\boldsymbol{F}_{\boldsymbol{0}} + \boldsymbol{A}\boldsymbol{y}) \tag{14}$$

Under the constraints:

$$\begin{gathered}g(\boldsymbol{y}) = \boldsymbol{F}_{\boldsymbol{m}\times\boldsymbol{1}} - (\boldsymbol{J}_{\boldsymbol{t}}^{\boldsymbol{T}})_{m\times n} \cdot (\boldsymbol{F}_{\boldsymbol{0}} + \boldsymbol{A}\boldsymbol{y}) = \boldsymbol{0} \\ r(\boldsymbol{y}) = \boldsymbol{F}_{\boldsymbol{min}} - (\boldsymbol{F}_{\boldsymbol{0}} + \boldsymbol{A}\boldsymbol{y}) \leq \boldsymbol{0}\end{gathered}$$

From Eq. (14) the problem of finding minimum norm positive tension actuator forces is reduced to minimizing a quadratic function with linear and first order constraints. To obtain the solution $\boldsymbol{y}$'s, you may apply Karush-Kuhn-Tucker Theorem. This is done by defining a function $\varepsilon(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ from the corresponding Lagrange multipliers $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \ldots \lambda_m]^T$ and $\boldsymbol{\mu} = [\mu_1, \mu_2 \ldots \mu_n]^T > 0$ as follows:

$$\varepsilon(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\boldsymbol{y}) + \boldsymbol{\lambda}^{\boldsymbol{T}}.g(\boldsymbol{y}) + \boldsymbol{\mu}^{\boldsymbol{T}}.r(\boldsymbol{y}) \qquad (15)$$

From Karush-Kuhn-Tucker Theorem, a necessary condition can be derived for $\boldsymbol{y_0}$ to yield to a local minimum of $f(\boldsymbol{y})$ under the conditions $g(\boldsymbol{y}) = 0$ and $r(\boldsymbol{y}) \leq 0$. This condition is $\mu^0 \geq 0$ $(\boldsymbol{\mu^0} = [\mu_1^0, \mu_2^0 \ldots \mu_n^0],\ \ \mu_i^0 \geq 0)$ that satisfies the following equations:

$$\frac{\partial}{\partial \boldsymbol{y}} \varepsilon(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})|_{\substack{y=y_0 \\ \lambda=\lambda_0 \\ \mu=\mu_0}} = \frac{\partial}{\partial \boldsymbol{y}} \big(f(\boldsymbol{y}) + \boldsymbol{\lambda}^{\boldsymbol{T}}.g(\boldsymbol{y}) + \boldsymbol{\mu}^{\boldsymbol{T}}.r(\boldsymbol{y})\big)|_{\substack{y=y_0 \\ \lambda=\lambda_0 \\ \mu=\mu_0}} = \boldsymbol{0} \qquad (16)$$

In which, $\boldsymbol{y_0}$ is a stationary point. Furthermore, the following condition must be also satisfied:

$$\boldsymbol{\mu}^{\boldsymbol{T}}.r(\boldsymbol{y})|_{y=y_0} = \boldsymbol{0}. \qquad (17)$$

Substitute $f(\boldsymbol{y})$, $g(\boldsymbol{y})$ and $r(\boldsymbol{y})$ into Eq.(15):

$$\begin{aligned}\varepsilon(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= \boldsymbol{F_0^T F_0} + \boldsymbol{F_0^T A y} + \boldsymbol{y^T A^T F_0} + \boldsymbol{y^T A^T A y} \\ &+ \boldsymbol{\lambda^T}(\boldsymbol{F_{m\times 1}} - \boldsymbol{J_t^T F_0}) - \boldsymbol{\lambda^T J_t^T A y} + \boldsymbol{\mu^T}(\boldsymbol{F_{min}} - \boldsymbol{F_0}) \\ &- \boldsymbol{\mu^T A y}\end{aligned} \qquad (18)$$

and differentiate the above equation and simply to:

$$\frac{\partial}{\partial \boldsymbol{y}} \varepsilon(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})|_{y=y_0} = 2\boldsymbol{F_0^T A} + 2\boldsymbol{y_0^T A^T A} - \boldsymbol{\lambda^T J_t^T A} - \boldsymbol{\mu^T A} = \boldsymbol{0}. \qquad (19)$$

Substitute $r(\boldsymbol{y})$ into Eq.(17):

$$\boldsymbol{\mu^T}(\boldsymbol{F_{min}} - \boldsymbol{F_0} - \boldsymbol{A y_0}) = \boldsymbol{0} \qquad (20)$$

Writing Eq. (19), $g(\boldsymbol{y}) = \boldsymbol{0}$ and Eq. (20) altogether we reach to:

$$\begin{cases} 2\,\boldsymbol{F_0^T A} + 2\boldsymbol{y_0^T}\,\boldsymbol{A^T A} - \boldsymbol{\lambda^T J_t^T A} - \boldsymbol{\mu^T A} = \boldsymbol{0} \\ \boldsymbol{F_{m\times 1}} - (\boldsymbol{J_t^T})_{\boldsymbol{m\times n}}.(\boldsymbol{F_0} + \boldsymbol{A y_0}) = \boldsymbol{0} \\ \boldsymbol{\mu^T}(\boldsymbol{F_{min}} - \boldsymbol{F_0} - \boldsymbol{A y_0}) = \boldsymbol{0} \end{cases} \qquad (21)$$

In this set of nonlinear equations $\boldsymbol{F_0}$ and $\boldsymbol{A}$ are known from Eq.(8) and Eq.(12), respectively. Note that the last equation of Eq. (21) is a nonlinear equation and it may lead to multiple solutions for this set of equations. By solving this set of equations $\boldsymbol{y_0}$ and $\boldsymbol{\mu^T}$ vectors are obtained. However, only the set of solution that satisfy the KKT theorem condition $\boldsymbol{\mu} = [\mu_1, \mu_2, \ldots \mu_n]^T \geq \boldsymbol{0}$ are acceptable.

If there is no set of solution with positive $\boldsymbol{\mu}$, the optimization problem does not lead to any solution. Moreover, it is well known that Karush-Kuhn-Tucker Theorem provides only the necessary condition to derive the local minimum of the optimization problem. In order to have sufficient condition for the solution, the convexity of the Lagrangian function $\varepsilon(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ must be analyzed. This analysis is done in the proceeding subsection.

B. *Lagrangian Function Convexity*

By substituting $g(\boldsymbol{y})$ and $r(\boldsymbol{y})$ into the Lagrangian function detailed in Eq.(15) and using the fact that $\boldsymbol{F_0}$ and $\boldsymbol{Ay}$ are orthogonal to each other, this function is simplified to:

$$\begin{aligned}\varepsilon(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= \boldsymbol{y^T A^T A y} + (\boldsymbol{F_0^T A} - \boldsymbol{\lambda^T J_t^T A} - \boldsymbol{\mu^T A})\boldsymbol{y} \\ &+ \boldsymbol{y^T A^T F_0} + \boldsymbol{F_0^T F_0} + \boldsymbol{\lambda^T}(\boldsymbol{F_{n\times 1}} - \boldsymbol{J_t^T F_0}) \\ &+ \boldsymbol{\mu^T}(\boldsymbol{F_{min}} - \boldsymbol{F_0})\end{aligned} \qquad (22)$$

Eq.(22) represents a quadratic form for the Lagrangian function $\varepsilon(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$. Convexities of quadratic functions are guaranteed provided that the second-order term of the quadratic form is positive definite. In such case, the Lagrangian function has a unique minimum. However, if this term is positive semi-definite, the quadratic function has an infinite number of local minimums. The second-order term of Eq. (22) is as follows:

$$\boldsymbol{A^T A} \qquad (23)$$

Thus sufficient condition for regular point $\boldsymbol{y_0}$ to be a minimum of the function is that $\boldsymbol{A^T A}$ is positive definite. As we explained earlier in Eq. (9) to (12), the matrix $\boldsymbol{A}$ is generated by collecting and orthonormalizing the linearly independent column vectors of $\boldsymbol{I_{n\times n}} - (\boldsymbol{J_t^T})^{\dagger}.\,\boldsymbol{J_t^T}$. Therefore, column vectors of $\boldsymbol{A}$ are linearly independent and each column vectors of $\boldsymbol{A}$ is orthogonal, and therefore:

$$\boldsymbol{A^T A} = \boldsymbol{I_{n\times n}} \qquad (24)$$

Therefore, $\boldsymbol{A^T A}$ is always positive definite, and the quadratic Lagrangian function given in Eq.(22) is always convex. Through this analysis the sufficient condition for regular point $\boldsymbol{y_0}$ to be a minimum is established.

III. ANALYTICAL SOLUTION OF THE OPTIMIZATION PROBLEM

In this subsection a procedure is given to generate the solution of the optimization problem in an analytical way. Note that $\varepsilon(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ is convex, $\boldsymbol{A^T A} = \boldsymbol{I}$ becomes positive definite and, therefore, nonsingular. Rewrite Eq.(21) by substituting $\boldsymbol{A^T A} = \boldsymbol{I}$ as:

$$\begin{cases} 2\boldsymbol{F_0^T A} + 2\boldsymbol{y_0^T} - \boldsymbol{\lambda^T J_t^T A} - \boldsymbol{\mu^T A} = \boldsymbol{0} \\ \boldsymbol{F_{m\times 1}} - \boldsymbol{J_t^T}.(\boldsymbol{F_0} + \boldsymbol{A y_0}) = \boldsymbol{0} \\ \boldsymbol{\mu^T}(\boldsymbol{F_{min}} - \boldsymbol{F_0} - \boldsymbol{A y_0}) = \boldsymbol{0} \end{cases} \qquad (25)$$

Use the transpose of the first equation:

$$\begin{cases} 2\boldsymbol{A^T F_0} + 2\,\boldsymbol{y_0} - \boldsymbol{A^T J \lambda} - \boldsymbol{A^T \mu} = \boldsymbol{0} \\ \boldsymbol{F_{m\times 1}} - \boldsymbol{J_t^T}.(\boldsymbol{F_0} + \boldsymbol{A y_0}) = \boldsymbol{0} \\ \boldsymbol{\mu^T}(\boldsymbol{F_{min}} - \boldsymbol{F_0} - \boldsymbol{A y_0}) = \boldsymbol{0} \end{cases} \qquad (26)$$

Eq. (17) is calculated as follows:

$$\sum_{i=1}^{n} \mu_i\, r_i(\boldsymbol{y_0}) = 0 \qquad (27)$$

Since, $\mu_i \geq 0$ and from Eq.(14) $r_i(\boldsymbol{y_0}) \leq 0$ Eq.(27) yields to

$$\begin{cases} \mu_i = 0 & for & r_i(\boldsymbol{y_0}) < 0 \\ \mu_i \geq 0 & for & r_i(\boldsymbol{y_0}) = 0 \end{cases} \qquad (28)$$

This equation implies that $\mu_i > 0$ only at instances where the following non-equality constraints hold $r_i(\boldsymbol{y}) = (\boldsymbol{F_{min}})_i - (\boldsymbol{\tau_{n\times1}})_i = 0$ or, $(\boldsymbol{\tau_{n\times1}})_i = (\boldsymbol{F_{min}})_i$. In fact, in this case the specific actuator force lies at the boundary of the non-equality constraint and should be set by the constant $(\boldsymbol{F_{min}})_i$. Furthermore, the condition $\mu_i = 0$ is satisfied only when the non-equality constraints $r_i(\boldsymbol{y}) = (\boldsymbol{F_{min}})_i - (\boldsymbol{\tau_{n\times1}})_i < 0$, or in other words the actuator force lies inside the solution set defined by this non-equality constraint. In other words, considering these facts, the solution of the optimization problem can be derived from three different cases:

Case 1:

Assume that all forces are inside the solution set defined by the non-equality constraint and $\mu_i = 0$. Hence, in this case $r_i(\boldsymbol{y}) > 0$ for all $i = 1, 2, \ldots, n$, and Eq.(26) is simplified to:

$$\begin{cases} 2\,\boldsymbol{A^T F_0} + 2\,\boldsymbol{y_0} - \boldsymbol{A^T J \lambda} = \boldsymbol{0} \\ \boldsymbol{F_{m\times1}} - \boldsymbol{J_t^T} \,.\, (\boldsymbol{F_0} + \boldsymbol{A y_0}) = \boldsymbol{0} \end{cases} \tag{29}$$

Eq.(29) can be written as a set of linear equations in the following matrix form:

$$[\boldsymbol{B_0}]_{(n+m)\times(n+m)} . [\boldsymbol{X_0}]_{(n+m)\times1} = [\boldsymbol{C_0}]_{(n+m)\times1} \tag{30}$$

In which,

$$\boldsymbol{B_0} = \begin{bmatrix} -2\,\boldsymbol{I_{n\times n}} & [\boldsymbol{A^T J}]_{n\times m} \\ [\boldsymbol{J_t^T A}]_{m\times n} & \boldsymbol{0}_{m\times m} \end{bmatrix}, \boldsymbol{X_0} = \begin{bmatrix} \boldsymbol{y_0}_{n\times1} \\ \boldsymbol{\lambda}_{m\times1} \end{bmatrix},$$
$$\boldsymbol{C_0} = \begin{bmatrix} [2\boldsymbol{A^T F_0}]_{n\times1} \\ [\boldsymbol{F}_{m\times1} - \boldsymbol{J_t^T F_0}]_{m\times1} \end{bmatrix}. \tag{31}$$

These linear equations can be easily solved, and the amount of $\boldsymbol{y_0}$ and $\boldsymbol{\lambda}$ is obtained with the assumption applies in this case that $\boldsymbol{\mu} = [0, 0, 0, 0]^T$. Therefore,

$$\boldsymbol{\tau_{n\times1}} = \boldsymbol{F_0} + \boldsymbol{A y_0} \tag{32}$$

Case 2:

Assume that all forces are on the boundary of the non-equality constraints, and $\mu_i \geq 0$. Therefore, $r_i(\boldsymbol{y}) = 0$ for all $i = 1, 2, \ldots, n$, and thus the optimal actuator forces for all joints are obtained from:

$$\boldsymbol{\tau_{n\times1}} = \boldsymbol{F_0} + \boldsymbol{A y_0} = \boldsymbol{F_{min}} \tag{33}$$

In other words, we can simplify Eq.(26) as a linear set of equation as:

$$\begin{cases} 2\,\boldsymbol{A^T F_0} + 2\,\boldsymbol{y_0} - \boldsymbol{A^T J \lambda} - \boldsymbol{A^T \mu} = \boldsymbol{0} \\ \boldsymbol{J_t^T A y_0} = \boldsymbol{J_t^T}\,\boldsymbol{F_0} - \boldsymbol{F_{m\times1}} \\ \boldsymbol{A y_0} = \boldsymbol{F_{min}} - \boldsymbol{F_0} \end{cases} \tag{34}$$

which, can be rewritten in the following matrix form:

$$\begin{bmatrix} -2\,\boldsymbol{I_{n\times n}} & [\boldsymbol{A^T J_t}]_{n\times m} & [\boldsymbol{A^T}]_{n\times n} \\ [\boldsymbol{J_t^T A}]_{m\times n} & [\boldsymbol{0}]_{m\times m} & [\boldsymbol{0}]_{m\times n} \\ [\boldsymbol{A}]_{n\times n} & [\boldsymbol{0}]_{n\times m} & [\boldsymbol{0}]_{n\times n} \end{bmatrix} . \begin{bmatrix} \boldsymbol{y_0}_{\boldsymbol{n\times1}} \\ \boldsymbol{\lambda_{m\times1}} \\ \boldsymbol{\mu_{n\times1}} \end{bmatrix}$$
$$= \begin{bmatrix} [2\boldsymbol{A^T F_0}]_{n\times1} \\ [\boldsymbol{F_{m\times1}} - \boldsymbol{J_t^T F_0}]_{m\times1} \\ [\boldsymbol{F_{min}} - \boldsymbol{F_0}]_{n\times1} \end{bmatrix} \tag{35}$$

or,

$$\begin{bmatrix} [\quad \boldsymbol{B_0} \quad]_{(n+m)\times(n+m)} & \begin{bmatrix} [\boldsymbol{A^T}]_{n\times n} \\ [\boldsymbol{0}]_{m\times n} \end{bmatrix}_{(n+m)\times n} \\ [[\boldsymbol{A}]_{n\times n} \quad [\boldsymbol{0}]_{n\times m}]_{(n+m)\times(n+m)} & [\quad \boldsymbol{0} \quad]_{n\times n} \end{bmatrix} \begin{bmatrix} y_{0_{n\times1}} \\ \lambda_{m\times1} \\ \mu_{n\times1} \end{bmatrix}$$
$$= \begin{bmatrix} [2\boldsymbol{A^T F_0}]_{n\times1} \\ [\boldsymbol{F_{m\times1}} - \boldsymbol{J_t^T F_0}]_{m\times1} \\ [\boldsymbol{F_{min}} - \boldsymbol{F_0}]_{n\times1} \end{bmatrix} \tag{36}$$

and

$$[\boldsymbol{B}]_{(2n+m)\times(2n+m)} . [\boldsymbol{X}]_{(2n+m)\times1} = [\boldsymbol{C}]_{(2n+m)\times1} \tag{37}$$

in which,

$$B = \begin{bmatrix} \boldsymbol{B_0} & \begin{bmatrix} \boldsymbol{A^T} \\ \boldsymbol{0} \end{bmatrix} \\ [\boldsymbol{A} \quad \boldsymbol{0}] & \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{X} = \begin{bmatrix} \boldsymbol{y_0} \\ \boldsymbol{\lambda} \\ \boldsymbol{\mu} \end{bmatrix}, and$$
$$\boldsymbol{C} = \begin{bmatrix} 2\boldsymbol{A^T F_0} \\ \boldsymbol{F_{m\times1}} - \boldsymbol{J_t^T F_0} \\ \boldsymbol{F_{min}} - \boldsymbol{F_0} \end{bmatrix} \tag{38}$$

By solving these linear equations all the following unknowns: $\boldsymbol{y_0}, \boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, can be easily obtained.

Case 3:

Consider the case, in which for the optimal solution $\mu_i > 0$, for some $i's$ within $[1,2,\ldots,n]$, and $\mu_j = 0$ for the rest of them $j \neq i$ and $j = [1,2,\ldots,n]$. In this case for the elements in which $\mu_i > 0$, the corresponding forces $\tau_i's$ are obtained from:

$$\tau_i = [\boldsymbol{F_0} + \boldsymbol{A y_0}]_i = [\boldsymbol{F_{min}}]_i, \tag{39}$$

and for the rest of μ_j's their corresponding forces are calculated from an equation deduced from Eq.(38), by eliminating the rows and columns of matrix B and corresponding elements of vectors $\boldsymbol{X}$ and $\boldsymbol{C}$ related to the zero μ_i's. Therefore, $\boldsymbol{y_0}, \boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ can be obtained by solving the following linear equations:

$$\begin{bmatrix} \boldsymbol{B_0} & \begin{bmatrix} [\vec{\boldsymbol{a}}_1^T & \cdots & \vec{\boldsymbol{a}}_j^T] \\ & \boldsymbol{0} & \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} \vec{\boldsymbol{a}}_1 \\ \vdots \\ \vec{\boldsymbol{a}}_j \end{bmatrix} & \boldsymbol{0} \end{bmatrix} & \boldsymbol{0} \end{bmatrix} . \begin{bmatrix} \boldsymbol{y_0} \\ \boldsymbol{\lambda} \\ \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_j \end{bmatrix} \end{bmatrix}$$
$$= \begin{bmatrix} 2\boldsymbol{A^T F_0} \\ \boldsymbol{F_{m\times1}} - \boldsymbol{J^T F_0} \\ \begin{bmatrix} \boldsymbol{F}_{min_1} - \boldsymbol{F}_{\boldsymbol{0}_1} \\ \vdots \\ \boldsymbol{F}_{min_j} - \boldsymbol{F}_{\boldsymbol{0}_j} \end{bmatrix} \end{bmatrix} \tag{40}$$

In this equation we suppose that $\boldsymbol{A} = \begin{bmatrix} \vec{\boldsymbol{a}}_1 \\ \vdots \\ \vec{\boldsymbol{a}}_n \end{bmatrix}_{n\times1}$, and each $\vec{\boldsymbol{a}}_j$ is a row vector, and only the row the vectors corresponding to the nonzero $\mu_i's$ are left in this equation. Finally, assuming that $\boldsymbol{F_{min}} \geq 0$ and this fact that $\varepsilon(\boldsymbol{y}, \boldsymbol{\lambda}, \boldsymbol{\mu})$ is convex, there is a unique minimum solution for this problem that can be found from the above mentioned three cases. The unique solution can therefore, be found through a search procedure which is detailed in the following subsection.

Search Algorithm:

This section is devoted to develop a search routine for the proposed algorithm given for the redundancy resolution solution. In this search routine and in the first loop, we assume that all forces lie inside the solution set defined by the non-equality constraints, i.e. $\forall\, \mu_i > 0$, and therefore, $\boldsymbol{y_0}$ is simply obtained from the solution given in case 1. If this solution satisfy all the optimization conditions, i.e. $r_i(\boldsymbol{y}) \leq 0\ \forall i$, then it is a valid solution for the optimization problem and the search algorithm is terminated. Otherwise, the combinations of forces that can lie on the boundaries of the non-equality constraints must be checked and found. For this search there are $\binom{m}{S}$ combinations with $m = [1,2,..n]^T$ and $S = 1 \text{ to } n$, which might be plausible solution for the problem. These solutions can be checked by sweeping all possible combinations through changing S in a loop. However, only a solution to the optimization problem is acceptable that satisfies all the optimization constraints i.e. $r_i(\boldsymbol{y}) \leq 0,\ \forall i$. Note that in this algorithm we solve the set of equations corresponding to KKT Theorem condition ($\forall\, \mu_i \geq 0$), and there is no need to re-check this condition. Let us summarize the search algorithm as follows. Assume $\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \dots \lambda_m]^T$, $\boldsymbol{\mu} = [\mu_1, \mu_2, \dots, \mu_n]$,

Step 0.

Set $S = 0$.

Step 1.

Assume S forces are on the boundaries of the non-equality constraints ($\mu_i = 0$) and $n - S$ forces are inside the solution set ($\mu_i > 0$). Find the possible compositions of the forces that lie on the boundaries of the non-equality constraints by:

compositions $\binom{\boldsymbol{M}}{S}$ and $\boldsymbol{M} = [1,2, \dots, n]^T$

Step 2.

For each combination:

a) For each i that $\mu_i = 0$ compute $\boldsymbol{y_0}$ and corresponding forces from:

$$\tau_i = [\boldsymbol{F_{min}}]_i$$

b) For each j that $\mu_j > 0$ eliminate the rows and columns of matrix $\boldsymbol{B}$ and corresponding elements of vectors $\boldsymbol{X}$ and , $\boldsymbol{C}$ related to the zero of $\mu_i{}'s$ and compute $\boldsymbol{y_0}$, and the rest of μ_j's and $\boldsymbol{\lambda}$ by solving the linear equations given in Eq.(40). Then compute corresponding forces as:

$$\tau_j = [\boldsymbol{F_0} + \boldsymbol{A}\boldsymbol{y_0}]_j$$

Augment $\mu_i's$ and μ_j's to generate μ, and consider the set of solution $[\boldsymbol{y_0}, \boldsymbol{\lambda}, \boldsymbol{\mu}]$.

Step 3:

Check if all derived μ_j's that satisfy $\mu_i \geq 0$, are also satisfying $r_i(\boldsymbol{y}) \leq 0$ condition. If this is true this set of solution is the optimal solution, and stop the search algorithm, otherwise continue.

Step 4:

If $S < m$ then set $S = S + 1$, and *go to* Step 1.

Step 5:

If $S = m$ then the optimization problem do not have any solution and the resultant forces $\boldsymbol{\tau_{n\times 1}}$ cannot be generated under these constraints. Finally, we can represent this solution as an optimal projection map that projects forces from Cartesian space into joint space.

IV. Implementation Results

In this section the developed redundancy resolution technique is implemented on a cable-driven redundant manipulator, and its performance in closed-loop control system is evaluated. The Block diagram of closed-loop simulations is given in Fig.1. As it is illustrated in this block diagram, the macro-micro cable-driven manipulator analyzed thoroughly in [18], is used as the manipulator under consideration. This manipulator has six degrees of freedom and two degrees of redundancy ($n = 6$ and $m = 8$). Furthermore, different redundancy resolution schemes, including the proposed analytical method and three other numerical methods are used as the redundancy resolution scheme. For this comparison, we used Matlab *fmincon* function that allows applying these numerical-iterative methods. The controller used in these simulations is composed of a decentralized PD controller for manipulator in addition to an inverse dynamics control. This control method is capable to provide the required tracking, despite the actuator saturation limits [18]. Fig.2 and Fig.3 illustrate the elapsed time required to calculate redundancy resolution scheme through different methods. The numerical methods used in here are, Trust-Region-Reflective [19], Active-Set [20], and Interior-Point Optimization [21].

As it is seen in these figures the time required to perform our proposed method is much less than that of the other numerical optimization methods. Furthermore, in the numerical algorithms simulated for this case the elapsed time is significantly varying in different iterations, and in some instances there is abrupt change in their variation. This is due to the fact that in these algorithms the number of iterations greatly depends to the robot configurations. In order to quantitatively compare different routines, Table 1 shows the total elapsed time used in different redundancy resolution routines. To track a desired 600-second trajectory, it is seen that the total elapsed time in our proposed analytical method is about 1.06 seconds, which is more than 41 times shorter than the interior-point method, and is the fastest scheme among all analyzed routines. This benefit is much more appreciated comparing the result with the trust-region-reflective optimization method, which is more than 71 times slower than our proposed method.

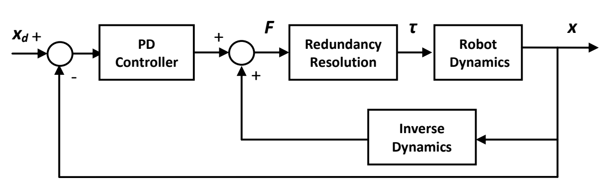

Figure 1. Block diagram of the closed-loop control system using inverse dynamics control in addition to decentralized PD controller.

V. Conclusions

In this paper an iterative-analytic solution for the redundancy resolution problem is proposed and its implementation on a cable-driven redundant manipulator is studied in detail. This task is formulated into an optimization problem with equality and non-equality constraints. Nonlinear programming techniques and particularly Karush-Kuhn-Tucker theorem is used to analyze this optimization problem and to generate an analytic solution. Subsequently, a tractable search algorithm is proposed to effectively generate such solutions. It is shown through simulations that, the elapsed time of the generated routine which implements the iterative-analytic redundancy resolution scheme in closed-loop structure is considerably less than that of other numerical and iterative optimization methods.

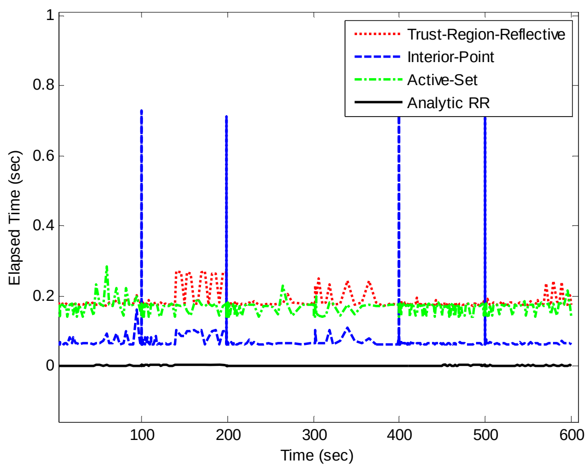

Figure 2. Elapsed time to calculate optimal forces in iterative-analytical redundancy resolution and other numerical optimizations methods at each iteration.

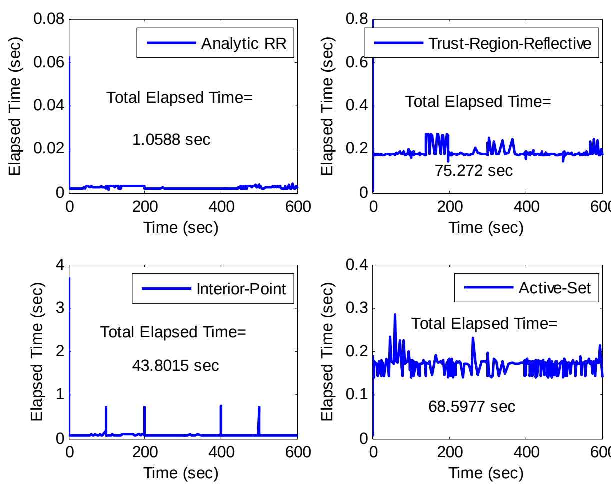

Figure 3. Total elapsed time to calculate optimal forces in iterative-analytical redundancy resolution and other numerical optimizations methods in each iteration.

Table 1: Total Elapsed Time to Execute The Redundancy Resolution Schemes

Algorithms	Total Elapsed Time (sec)	Speed Comparison
Proposed Method	1.0588	1
Trust-Region-Reflective	75.272	71.0918
Interior-Point	43.8015	41.3690
Active-Set	68.5977	64.7882

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