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Delay-Dependent H_∞ Control of Linear Systems with input Delay Using Composite State-Derivative Feedback

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***Abstract*— H_∞ control problem for input-delayed systems is considered in this paper. A composite state-derivative control law is used, in which, a composition of the state variables and their derivatives appear in the control law. Thus, the resulting closed-loop system turns into a specific time-delay system of neutral type. The significant specification of this neutral system is that its delayed term coefficients depend on the control law parameters. This condition provides new challenging issues which has its own merits in theoretical research as well as application aspects. New delay-dependent sufficient condition for the existence of H_∞ controller in terms of matrix inequalities is derived in the present paper. The resulting H_∞ controller guarantees asymptotic stability of the closed-loop system as well as a guaranteed limited H_∞ norm smaller than a prescribed level. Numerical examples are presented to illustrate the effectiveness of the proposed methods.**

I. Introduction

In various engineering systems such as chemical processes, rolling mills and nuclear reactors, time-delays are frequently a source of instability. Hence, many researchers have paid great attention to the stabilization and control of time-delay systems of retarded or neutral type.

Referring to H_∞ control of neutral systems, robust H_∞ state feedback control of uncertain neutral system has been considered in [1]. An optimization problem has been formulated with linear matrix inequality constraints to obtain an H_∞ state feedback controller. Observer-based H_∞ state feedback control for a class of uncertain neutral systems is another topic which Lien has considered in [2]. H_∞ output feedback control of neutral systems has also been the centre of attention in some research such as [3] and [4]. Moreover, an H_∞ output feedback controller has been designed in terms of three LMIs using bounded real lemma in [3], for a neutral system with multiple delays.

A general representation of a neutral system is shown as

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{n} A_{h_i} x(t-h_i) + \sum_{j=1}^{k} A_{d_j} \dot{x}(t-d_j) + Bu(t) \qquad (1)$$

To the best knowledge of the authors, in all developed theories for neutral systems, no analysis and synthesis has been derived when both $x(t-h_i)$ and $\dot{x}(t-h_i)$ s' coefficients are dependent on the controller's parameter, whereas these condition has its own merits in practical application as well as leading to a new challenging theoretical problem. The importance of the above condition is observed in the control systems such as active vibration suppression. Du and Zhang proposed a H_∞ state-feedback controller for an input-delay active suspension system [5]. Since the ride comfort is an important objective which is related to the body acceleration sensed by the passenger, an acceleration feedback can effectively improve this performance objective in active suspension system or generally, in active vibration suppression systems. Some researchers have paid considerable attention to this idea such as [6] and [7]. Abdelaziz and válašek [7] proposed a formula similar to Ackermann for solving the pole-placement problem for non-delay linear single-input/single-output systems and multi-input/multi-output systems using state-derivative feedback. Assoncao et.al. [6] used this idea to design a stabilizing state-derivative controller for a delay-free system which bounds the output peak as well as the state-derivative feedback. Moreover, an analysis for the stability of a system controlled by composite state-derivative feedback in presence of small uncertain delays in the feedback loop was presented in [8]. To the best of our knowledge, no synthesis of state-derivative feedback has been presented for input-delay systems in the literature, whereas, as described earlier, it could be of great significance in practice.

To benefit the advantages of the state feedback as well as acceleration feedback or generally, state derivative feedback, we employ composite control law as it is shown by the following equation:

$$u = K_1 x + K_2 \dot{x} \qquad (2)$$

Assume the general representation of linear input-delayed systems as follows:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{n} B_i u(t-h_i) + Ew(t) \qquad (3)$$

Applying the control law (2) to the input time-delay system (3) leads to a time delay closed system of neutral type which is represented as follows:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{n} B_i K_1 x(t-h_i) + \sum_{i=1}^{n} B_i K_2 \dot{x}(t-h_i) + Ew(t) \qquad (4)$$

As it is seen in the Equation (4), both $x(t-h_i)$ and $\dot{x}(t-h_i)$ s' coefficients are functions of the control law parameters. Therefore, finding K_1 and K_2 introduces a new challenging problem theoretically, whereas the choice of K_1 and K_2 can be very effective in obtaining desired performance in such applications. The main purpose of this paper is to elaborate this problem in detail and to design H_∞-based controller for the closed-loop system.

This paper is organized as follows. Problem formulation is introduced in Section 2, and in Section 3, an H_∞ controller is designed in terms of some matrix inequalities for the closed-loop time-delay system of neutral type. Illustrative examples are provided in section 4 to show the effectiveness of the proposed methods in some case studies, and real application. Finally, the concluding remarks are given in Section 5.

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II. PROBLEM FORMULATION

In this paper, we consider the following time-delay system with input delay:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t-\tau) + Ew(t) \\ z(t) &= Cx(t) + D_1u(t-\tau) + D_2w(t)\end{aligned} \tag{5}$$

where x is the state, $w\in\Re^p$ is the disturbance input of system that belongs to $L_2[0,\infty)$, τ is the constant time-delay of the system and is assumed to satisfy $0<\tau\le\bar{\tau}$, $u\in\Re^m$ is the system input and $z\in\Re^q$ is the controlled system output. The matrices $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$, $E\in\Re^{n\times p}$, $C\in\Re^{q\times n}$, $D_1\in\Re^{q\times m}$, $D_2\in\Re^{q\times p}$ are assumed to be known. In this paper we assume that all the state variables are measured. Considering the composite control law (2), the state space equations of the closed-loop system is given by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + BK_1x(t-\tau) + BK_2\dot{x}(t-\tau) + Ew(t) \\ z(t) &= Cx(t) + D_1K_1x(t-\tau) + D_1K_2\dot{x}(t-\tau) + D_2w(t) \\ x(t_0+\theta) &= \phi(\theta) \qquad \forall\theta\in[-\tau,0]\end{aligned} \tag{6}$$

Therefore, the resulting closed-loop system (6) is a time-delay system of neutral type which both coefficients of $x(t-\tau)$ and $\dot{x}(t-\tau)$ depending on the controller parameters. Here, we state two well known lemmas which will be used further in the main result of the paper.

Lemma 1 [9]: Assume $a(.)\in\Re^{na}$, $b(.)\in\Re^{nb}$ and $N\in\Re^{na\times nb}$ are defined on the interval Ω, then for any matrices $X\in\Re^{na\times nb}$, $Y\in\Re^{na\times nb}$ and $Z\in\Re^{na\times nb}$, the following holds:

$$-2\int_\Omega a^T(\alpha)Nb(\alpha)d\alpha \le \int_\Omega \begin{bmatrix} a(\alpha) \\ b(\alpha)\end{bmatrix}^T \begin{bmatrix} X & Y-N \\ Y^T-N^T & Z\end{bmatrix}\begin{bmatrix} a(\alpha) \\ b(\alpha)\end{bmatrix} d\alpha,$$

Where $\begin{bmatrix} X & Y \\ Y^T & Z\end{bmatrix} > 0$

Remark 1: The above inequality can be extended to the similar inequalities with multiple integrals.

Lemma 2 [10]: (Schur Complement) The LMI $\begin{bmatrix} Q(y) & S(y) \\ S^T(y) & R(y)\end{bmatrix} < 0$ is equivalent to

$$R(y) < 0, \quad Q(y) - S(y)R(y)^{-1}S(y)^T < 0,$$

Lemma 3 [11]: For a prescribed matrix, $M = \begin{bmatrix} A \\ B\end{bmatrix}$ or $M = \begin{bmatrix} A & B\end{bmatrix}$ we have the following inequality:

$$\max\{\bar{\sigma}(A), \bar{\sigma}(B)\} \le \bar{\sigma}(M) \le \sqrt{2}\max\{\bar{\sigma}(A), \bar{\sigma}(B)\}$$

III. H_∞ CONTROL DESIGN

In many developed theories, conventional state feedback controller has been used for obtaining stability as well as performance objectives of the closed-loop system. In spite of the effectiveness of state feedback controller in many applications, it is not suitable for the cases that we need to have an acceleration feedback or generally derivative of the state in the feedback. On the other hand, H_∞ control is an effective method which guarantees asymptotic stability as well as performance objectives. This is why H_∞ control for time-delay systems has been among the most challenging topics in recent years. All the aforementioned facts motivate us to elaborate on the following Theorem and one Lemma which are stated in this section.

Theorem 1: Given scalar $\bar{\tau}>0$, the closed-loop system (6) is asymptotically stable and $\|T_{zw}\|_\infty<\gamma$ for any constant time-delay τ satisfying $0<\tau\le\bar{\tau}$, if there exist positive definite symmetric matrices L, T, H_1, H_2, F_1, $F_2\in\Re^{n\times n}$, negative definite symmetric matrix N_2 and matrices M_1, M_2, N_1, $\in\Re^{n\times n}$, V, $W\in\Re^{m\times n}$ satisfying matrix inequalities (7) ~ (9). Moreover, H_∞ composite state-derivative feedback control law is given by $u = VL^{-1}x + WL^{-1}\dot{x}$.

$$\begin{bmatrix} M_1 & N_1 \\ N_1^T & LF_1^{-1}L\end{bmatrix} > 0 \tag{8}$$

and,

$$\begin{bmatrix} 2M_2 & \bar{\tau}N_2 \\ \bar{\tau}N_2^T & 2LF_2^{-1}L\end{bmatrix} > 0 \tag{9}$$

in which,

$$\Omega = LA^T + AL + N_1 + N_1^T + \bar{\tau}(M_1+M_2) + (2-\bar{\tau})N_2 + T$$

First, let us prove following useful lemma which will be applied in the proof of Theorem 1.

Lemma 4: Consider the neutral system (6) and assume $d(t)=[\,w^T(t) \quad \dot{w}^T(t)\,]^T$. If $\|T_{zd}\|_\infty<\gamma$, then the inequality $\|T_{zw}\|_\infty<\gamma$ is satisfied.

Proof: Since we have $d(s) = \begin{bmatrix} w(s) \\ sw(s)\end{bmatrix}$, then the transfer function from d to z can be expressed by

$$T_{zd}(s) = \left[\left(z(s)/w(s)\right)^T \quad \left(z(s)/sw(s)\right)^T\right]^T$$

$$\begin{bmatrix}
\Omega & BV_1-N_1-N_2 & BW-\bar{\tau}N_2 & 0 & E+LC^TD_2 & 0 & \bar{\tau}LA^T & LA^T & \bar{\tau}LA^TA^T & LA^TA^T & LC^T \\
* & -T & 0 & 0 & (D_1V)^TD_2 & 0 & \bar{\tau}(BV)^T & (BV)^T & \bar{\tau}(ABV)^T & (ABV)^T & (D_1V)^T \\
* & * & -LH_1^{-1}L-\bar{\tau}N_2 & 0 & (D_1W)^TD_2 & 0 & \bar{\tau}(BW)^T & (BW)^T & \bar{\tau}(BV+ABW)^T & (BV+ABW)^T & (D_1W)^T \\
* & * & * & -LH_2^{-1}L & 0 & 0 & 0 & 0 & \bar{\tau}(BW)^T & (BW)^T & 0 \\
* & * & * & * & D_2^TD_2-\gamma^2I & 0 & \bar{\tau}E^T & E^T & \bar{\tau}E^TA^T & E^TA^T & 0 \\
* & * & * & * & * & -\gamma^2I & 0 & 0 & \bar{\tau}E^T & E^T & 0 \\
* & * & * & * & * & * & -\bar{\tau}F_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -H_1 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\bar{\tau}F_2 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -H_2 & 0 \\
* & * & * & * & * & * & * & * & * & * & -I
\end{bmatrix} < 0 \tag{7}$$

The above equality can be rewritten as

$$T_{zd}(s)=\left[T_{zw}^T(s)\quad \frac{1}{s}T_{zw}^T(s)\right]^T$$

By the Lemma 3, the following inequality holds as

$$\max\left\{\left\|T_{zw}\right\|_\infty \ ,\ \left\|\frac{1}{s}T_{zw}\right\|_\infty\right\}\le\left\|\begin{bmatrix}T_{zw}(s)\\ \frac{1}{s}T_{zw}(s)\end{bmatrix}\right\|_\infty$$

By the above inequality, it can be easily concluded that $\left\|\begin{bmatrix}T_{zw}(s)\\ \frac{1}{s}T_{zw}(s)\end{bmatrix}\right\|_\infty<\gamma$ guarantees $\left\|T_{zw}\right\|_\infty<\gamma$. This completes the proof. €

Corollary 1: Consider the neutral system (6) and two following performance indices:

$$J_1(w)=\int_0^\infty\left(z^Tz-\gamma^2w^Tw\right)d\tau,\ J_2(w)=\int_0^\infty(z^Tz-\gamma^2d^Td)d\tau$$

Where $d(t)=[\,w^T(t)\quad \dot{w}^T(t)\,]^T$. Since the inequalities $J_1<0$ and $J_2<0$ corresponds to H_∞ constraints $\left\|T_{zw}\right\|_\infty<\gamma$ and $\left\|T_{zd}\right\|_\infty<\gamma$ respectively, then for the inequality $J_1<0$ to be satisfied, it suffices to show that the condition $J_2<0$ is satisfied.

Proof of Theorem 1: In this case a Lyapunov-Krasovskii functional candidate for the system (6) has the form

$$V=V_1+V_2+V_3+V_4$$

Where

$$V_1=x(t)^TPx(t) \tag{10}$$

$$V_2=\int_{-\tau}^0\int_{t+\beta}^t\dot{x}^T(\alpha)Z_1\dot{x}(\alpha)\,d\alpha\,d\beta, \tag{11}$$

$$V_3=\int_{t-\tau}^t x^T(\alpha)Qx(\alpha)d\alpha+\int_{t-\tau}^t\dot{x}^T(\alpha)R_1\dot{x}(\alpha)d\alpha \tag{12}$$

$$\begin{aligned}V_4=&\tau^{-1}\int_{-\tau}^0\int_{-\tau}^\beta\int_{t+\eta}^t\ddot{x}^T(\alpha)Z'\ddot{x}(\alpha)d\alpha d\eta d\beta\\&+(1/2)\int_{t-\tau}^t\ddot{x}^T(\alpha)R'\ddot{x}(\alpha)d\alpha,\end{aligned} \tag{13}$$

where $P=P^T>0$, $Q=Q^T>0$, $R_1=R_1^T>0$, $R'=R'^T>0$, $Z_1=Z_1^T>0$ and $Z'=Z'^T>0$. Differentiating V_1 with respect to t gives us

$$\begin{aligned}\dot{V}_1&=2x^T(t)P\dot{x}(t)\\&=2x^T(t)P\{Ax(t)+BK_1x(t-\tau)+BK_2\dot{x}(t-\tau)+Ew(t)\}\end{aligned}$$

It is possible to write

$$x(t-\tau)=x(t)-\int_{t-\tau}^t\dot{x}(\alpha)d\alpha \tag{14}$$

We introduce the following relation for the delayed derivative of the state:

$$\dot{x}(t-\tau)=\tau^{-1}\left[x(t)-x(t-\tau)-\int_{-\tau}^0\int_{t-\tau}^{t+\beta}\ddot{x}(\alpha)d\alpha d\beta\right] \tag{15}$$

Therefore,

$$\begin{aligned}\dot{V}_1=&2x^TP(A+BK_1+\tau^{-1}BK_2)x(t)\\&-2\tau^{-1}x^TPBK_2x(t-\tau)-2x^TPBK_1\int_{t-\tau}^t\dot{x}(\alpha)d\alpha\\&-2\tau^{-1}x^TPBK_2\int_{-\tau}^0\int_{t-\tau}^{t+\beta}\ddot{x}(\alpha)d\alpha d\beta+2x^T(t)PEw(t)\end{aligned}$$

Applying Lemma 2 and Remark 1, the following upper bound for $\dot{V}_1$ is obtained:

$$\begin{aligned}\dot{V}_1\le& x^T\{A^TP+PA+\tau\left(X_1+X'/2\right)+Y_1+Y_1^T+\tau^{-1}\left(Y'+Y'^T\right)\}x\\&+x^T(t)(PBK_1-Y_1)x(t-\tau)+x^T(t-\tau)(PBK_1-Y_1)^Tx(t)\\&+x^T(t)(PBK_2-Y')\dot{x}(t-\tau)+\dot{x}^T(t-\tau)(PBK_2-Y')^Tx(t)\\&-\tau^{-1}x^T(t)Y'x(t-\tau)-\tau^{-1}x^T(t-\tau)Y'^Tx(t)+2x^T(t)PEw(t)\\&+\tau^{-1}\int_{-\tau}^0\int_{t-\tau}^{t+\beta}\ddot{x}^T(\alpha)Z'\ddot{x}(\alpha)d\alpha d\beta+\int_{t-\tau}^t\dot{x}^T(\alpha)Z_1\dot{x}(\alpha)d\alpha\end{aligned} \tag{16}$$

where

$$\begin{bmatrix}X_1 & Y_1\\ Y_1^T & Z_1\end{bmatrix}>0 \tag{17}$$

and,

$$\begin{bmatrix}X' & Y'\\ Y'^T & Z'\end{bmatrix}>0 \tag{18}$$

Also, the time derivative of V_4 can be represented as follows:

$$\begin{aligned}\dot{V}_4=&-\tau^{-1}\int_{-\tau}^0\int_{t-\tau}^{t+\beta}\ddot{x}^T(\alpha)Z'\ddot{x}(\alpha)d\alpha d\beta+(\tau/2)\ddot{x}^T(t)Z'\ddot{x}(t)\\&+(1/2)\ddot{x}^T(t)R'\ddot{x}(t)-(1/2)\ddot{x}^T(t-\tau)R'\ddot{x}(t-\tau)\end{aligned} \tag{19}$$

It can be shown that the time derivative of V_2 and V_3 are

$$\dot{V}_2=\dot{x}^T(t)(\tau Z_1)\dot{x}(t)-\int_{t-\tau}^t\dot{x}^T(\alpha)Z_1\dot{x}(\alpha)d\alpha \tag{20}$$

$$\begin{aligned}\dot{V}_3=&x^T(t)Qx(t)-x^T(t-\tau)Qx(t-\tau)\\&+\dot{x}^T(t)R_1\dot{x}(t)-\dot{x}^T(t-\tau)R_1\dot{x}(t-\tau)\end{aligned} \tag{21}$$

Therefore we have

$$\dot{V}(t)=\sum_{i=1}^4\dot{V}_i \tag{22}$$

We consider (16), (19) ~ (22) and define $\tau^{-1}Y'=Y_2$. By the assumption of $Y'=Y'^T<0$ and then adding and subtracting the terms $x^T(t)(\tau Y_2)x(t)$ and $\dot{x}^T(t-\tau)(\tau Y_2)^T\dot{x}(t-\tau)$ in (22), an upper bound for $\dot{V}$ is obtained as

$$\begin{aligned}\dot{V}(t)\le& x^T\{A^TP+PA+\tau\left(X_1+(1/2)X'\right)+Y_1+Y_1^T+2Y_2\}x\\&+2x^T(t)(PBK_1-Y_1)x(t-\tau)+2x^T(t)PBK_2\dot{x}(t-\tau)\\&+\left(x^T(t)+\dot{x}(t-\tau)\right)^T(-\tau Y_2)\left(x^T(t)+\dot{x}(t-\tau)\right)+2x^T(t)PEw(t)\\&-x^T(t)Y_2x(t-\tau)-x^T(t-\tau)Y_2x(t)+(\tau/2)\ddot{x}^T(t)Z'\ddot{x}(t)\\&+(1/2)\ddot{x}^T(t)R'\ddot{x}(t)-(1/2)\ddot{x}^T(t-\tau)R'\ddot{x}(t-\tau)\\&+\left(Ax+BK_1x(t-\tau)+BK_2\dot{x}(t-\tau)+Ew(t)\right)^T\tau Z_1.\\&\left(Ax+BK_1x(t-\tau)+BK_2\dot{x}(t-\tau)+Ew(t)\right)+x^T(t)Qx(t)\\&-x^T(t-\tau)Qx(t-\tau)+\dot{x}^T(t)R_1\dot{x}(t)-\dot{x}^T(t-\tau)R_1\dot{x}(t-\tau)\\&+2\left(Ax(t)+BK_1x(t-\tau)+BK_2\dot{x}(t-\tau)+Ew(t)\right)^T(R_1)Ew(t)\\&+x^T(t)(\tau Y_2)x(t)\ +\dot{x}^T(t-\tau)(\tau(t)Y_2)^T\dot{x}(t-\tau)\end{aligned} \tag{23}$$

Assume zero initial condition, i.e. $\phi(t)=0$, $\forall t\in[-\tau,0]$ we have $V(q(t))|_{t=0}=0$. For a prescribed $\gamma>0$, consider the following performance index:

$$J_{zd}(w)=\int_0^{\infty}\left(z^T z-\gamma^2 d^T d\right)d\tau \qquad (24)$$

where $d(t)=\begin{bmatrix} w(t) \\ \dot{w}(t) \end{bmatrix}$,

Therefore the performance index (24) can be rewritten as

$$J_{zd}(w)=\int_0^{\infty}\left(z^T z-\gamma^2 w^T w-\gamma^2 \dot{w}^T \dot{w}\right)d\tau$$

Since $V(t)|_{t=0}=0$ and $V(t)|_{t\to\infty}\geq 0$, we obtain

$$\begin{aligned} J_{zd}(w) &= \int_0^{\infty}\left(z^T z-\gamma^2 w^T w-\gamma^2 \dot{w}^T \dot{w}+\dot{V}(t)\right)d\tau+V(t)\big|_{t=0}-V(t)\big|_{t\to\infty} \\ &\leq \int_0^{\infty}\left(z^T z-\gamma^2 w^T w-\gamma^2 \dot{w}^T \dot{w}+\dot{V}(t)\right)d\tau \end{aligned}$$

Hence the following inequality is obtained:

$$\begin{aligned} J_{zd}(w) \leq \int_0^{\infty} \{ & x^T C^T C x+2x^T C^T D_1 K_1 x(t-\tau)+2x^T C^T D_1 K_2 \dot{x}(t-\tau) \\ & +2x^T C^T D_2 w+x^T(t-\tau)K_1^T D_1^T D_1 K_1 x(t-\tau) \\ & +2x^T(t-\tau)K_1^T D_1^T D_1 K_2 \dot{x}(t-\tau)+2x^T(t-\tau)K_1^T D_1^T D_2 w \\ & +\dot{x}^T(t-\tau)K_2^T D_1^T D_1 K_2 \dot{x}(t-\tau)+2\dot{x}^T(t-\tau)K_2^T D_1^T D_2 w \\ & +w^T D_2^T D_2 w-\gamma^2 w^T w-\gamma^2 \dot{w}^T \dot{w}+\dot{V}(t)\}d\tau \end{aligned} \qquad (25)$$

Considering (23) and $0<\tau\leq\bar{\tau}$ a new upper bound for (25) is obtained as

$$J_{zd}\leq\int_0^{\infty}\left\{\zeta^T\Pi\zeta+x^T(t)(\tau Y_2)x(t)\ +\dot{x}^T(t-\tau)(\tau Y_2)^T\dot{x}(t-\tau)\right\}d\tau \qquad (26)$$

with defined

$$\zeta=\begin{bmatrix} x(t) & x(t-\tau) & \dot{x}(t-\tau) & \ddot{x}(t-\tau) & w(t) & \dot{w}(t) \end{bmatrix}$$

and $\Pi=\left[\Sigma_{ij}\right]$ where $\Sigma_{ij}=\Sigma_{ij}^T$ and $i,j=1,2,\ldots 6$.

in witch

$$\begin{aligned} \Sigma_{11} &= A^T P+PA+Y_1+Y_1^T+\bar{\tau}\left(X_1+X'/2\right)+(2-\bar{\tau})Y_2 \\ &\quad +A\Upsilon_1 A+A^T A^T\Upsilon_2 AA+Q+C^T C \\ \Sigma_{12} &= PBK_1-Y_1-Y_2+A^T\Upsilon_1 BK_1+A^T A^T\Upsilon_2 ABK_1+C^T D_1 K_1 \\ \Sigma_{13} &= PBK_2-\bar{\tau}Y_2+A^T\Upsilon_1 BK_2+A^T A^T\Upsilon_2\left(BK_1+ABK_2\right)+C^T D_1 K_2 \\ \Sigma_{14} &= A^T A^T\Upsilon_2 BK_2 \qquad \Sigma_{15}=PE+A^T\Upsilon_1 E+A^T A^T\Upsilon_2 AE+C^T D_2 \\ \Sigma_{16} &= A^T A^T\Upsilon_2 E \\ \Sigma_{22} &= -Q+\left(BK_1\right)^T\Upsilon_1 BK_1+\left(ABK_1\right)^T\Upsilon_2 ABK_1+K_1^T D_1^T D_1 K_1 \\ \Sigma_{23} &= \left(BK_1\right)^T\Upsilon_1 BK_2+K_1^T D_1^T D_1 K_2+\left(BK_1\right)^T A^T\Upsilon_2\left(BK_1+ABK_2\right) \\ \Sigma_{24} &= \left(BK_1\right)^T A^T\Upsilon_2 BK_2 \quad \Sigma_{25}=\left(BK_1\right)^T\Upsilon_1 E+\left(ABK_1\right)^T\Upsilon_2 AE+K_1^T D_1^T D_2 \\ \Sigma_{26} &= \left(ABK_1\right)^T\Upsilon_2 E \\ \Sigma_{33} &= -R_1-\bar{\tau}Y_2+\left(BK_2\right)^T\Upsilon_1 BK_2+K_2^T D_1^T D_1 K_2 \\ &\quad +\left(BK_1+ABK_2\right)^T\Upsilon_2\left(BK_1+ABK_2\right) \\ \Sigma_{34} &= \left(BK_1+ABK_2\right)^T\Upsilon_2 BK_2 \\ \Sigma_{35} &= \left(BK_2\right)^T\Upsilon_1 E+\left(ABK_2+BK_1\right)^T\Upsilon_2 AE+K_2^T D_1^T D_2 \\ \Sigma_{36} &= \left(BK_1+ABK_2\right)^T\Upsilon_2 E \qquad \Sigma_{44}=-R'/2+\left(BK_2\right)^T\Upsilon_2 BK_2 \\ \Sigma_{45} &= \left(BK_2\right)^T\Upsilon_2 AE \qquad \Sigma_{46}=\left(BK_2\right)^T\Upsilon_2 E \\ \Sigma_{55} &= E^T\Upsilon_1 E+E^T A^T\Upsilon_2 AE+D_2^T D_2-\gamma^2 I \\ \Sigma_{56} &= E^T A^T\Upsilon_2 E \qquad \Sigma_{66}=E^T\Upsilon_2 E-\gamma^2 I \end{aligned}$$

where $\Upsilon_1=R_1+\bar{\tau}Z_1$ and $\Upsilon_2=R'/2+(\bar{\tau}/2)Z'$.

Considering the constraint $Y_2=Y_2^T<0$, if $\zeta^T\Pi\zeta<0$, then the negative semi definiteness of J_{zd} in (26) is guaranteed for any constant time-delay τ satisfying $0<\tau\leq\bar{\tau}$. Hence, when assuming $w(t),\dot{w}(t)\in L_2[0\quad\infty)$ and $\Pi<0$ then implies that $J_{zd}<0$ and therefore $\|T_{zd}\|_\infty<\gamma$. This condition is the H_∞ performance to guarantee the tracking performance. By Lemma 4, the inequality $\|T_{zd}\|_\infty<\gamma$ guarantees $\|T_{zw}\|_\infty<\gamma$ to be satisfied. Using Schur complement, the condition $\Pi<0$ is equivalent to the following matrix inequality:

$$\left[\begin{array}{c|c} \Xi_{11} & \Xi_{12} \\ \hline \Xi_{12}^T & \Xi_{22} \end{array}\right]<0 \qquad (27)$$

with LMI (17) and

$$\begin{bmatrix} X' & \tau Y_2 \\ \tau Y_2 & Z' \end{bmatrix}>0 \qquad (28)$$

where

$$\Xi_{11}=\begin{bmatrix} \Omega_1 & PBK_1-Y_1-Y_2 & PBK_2-\bar{\tau}Y_2 & 0 & PE+C^T D_2 & 0 \\ * & -Q & 0 & 0 & K_1^T D_1^T D_2 & 0 \\ * & * & -R_1-\bar{\tau}Y_2 & 0 & K_2^T D_1^T D_2 & 0 \\ * & * & * & -R'/2 & 0 & 0 \\ * & * & * & * & D_2^T D_2-\gamma^2 I & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix}$$

$$\Xi_{12}=\begin{bmatrix} \bar{\tau}\,\Delta_1 Z_1 & \Delta_1 R_1 & \bar{\tau}\,\Delta_2 Z'/2 & \Delta_2 R'/2 & \Delta_3 \end{bmatrix}$$

where

$$\Omega_1=A^T P+PA+Y_1+Y_1^T+\bar{\tau}\left(X_1+(1/2)X'\right)+(2-\bar{\tau})Y_2+Q$$

$$\Delta_1=\begin{bmatrix} A & BK_1 & BK_2 & 0 & E & 0 \end{bmatrix}^T \qquad \Delta_3=\begin{bmatrix} C & D_1K_1 & D_1K_2 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\Delta_2=\begin{bmatrix} AA & ABK_1 & BK_1+ABK_2 & BK_2 & AE & E \end{bmatrix}^T$$

and

$$\Xi_{22}=diag\left(-\bar{\tau}Z_1 \quad -R_1 \quad -\bar{\tau}Z'/2 \quad -R'/2 \quad -I\right)$$

Denote $P^{-1}, Z_1^{-1}, R_1^{-1}$ as L, F_1, and H_1 respectively, by performing a congruence transformation to (27) by diag ($L, L, L, L, F_1, H_1, 2Z'^{-1}, 2R'^{-1}$) together with introducing the change of variables $M_1=LX_1L$, $M_2=L(X'/2)L$, $N_1=LY_1L$, $N_2=LY_2L$, $F_1=Z_1^{-1}$, $T=LQL$, $F_2=2Z'^{-1}$, $H_1=R_1^{-1}$, $H_2=2R'^{-1}$, $V=K_1L$, $W=K_2L$, the matrix inequality (7) is derived. Furthermore, pre and post multiplying the LMI (28) by diag (L, L) and its transpose and defining the same change of variables, the matrix inequality (8) is provided.

Similarly, by performing a congruence transformation to (28) by diag (L, L), we can obtain

$$\begin{bmatrix} LX'L & \tau LY_2L \\ \tau LY_2^T L & LZ'L \end{bmatrix}>0$$

Using Schur complement, we have

$$LX'L-\left(\tau LY_2L\right)\left(LZ'L\right)^{-1}\left(\tau LY_2^T L\right)>0$$

Substituting $M_2=L\left(X'/2\right)L$, $N_2=LY_2L$ and $F_2=2Z'^{-1}$, the following matrix inequality is derived.

$$2M_2-\left(\tau N_2\right)\left(2LF_2^{-1}L\right)^{-1}\left(\tau N_2^T\right)>0 \qquad (29)$$

On the other hand we have

$$2M_2-(\tau N_2)\left(2LF_2^{-1}L\right)^{-1}(\tau N_2)^T \geq 2M_2-(\bar{\tau} N_2)\left(2LF_2^{-1}L\right)^{-1}(\bar{\tau} N_2)^T \quad (30)$$

Therefore, satisfying the following inequality guarantees the inequality (29) to be satisfied.

$$2M_2-(\bar{\tau} N_2)\left(2LF_2^{-1}L\right)^{-1}(\bar{\tau} N_2)^T > 0 \quad (31)$$

Applying Schur complement, the matrix inequality (9) is obtained. This completes the proof. €

Remark 2: It is worth noting that in practical problems, the disturbance signals are usually differentiable therefore; in the proof of the Theorem 1 the constraint $\dot{w}(t)\in L^2[0\ \ \infty)$ is not a limiting condition.

Remark 3: It should be noted that generally, the problem of finding the smallest γ>0, namely γ_0, can be computed by solving the following optimization problem in L, T, H_1, H_2, F_1, F_2 >0, N_2<0 and $\sigma = \gamma^2$:

Minimize σ

Subject to L, T, H_1, H_2, F_1, F_2 >0, N_2<0, σ>0 and matrix inequality conditions (7) ~ (9)

Remark 4: Note that, the resulting conditions presented in the Theorem 1 are not LMI conditions. Gao and Wang [12] presented a modified algorithm using Moon's idea to find a minimum noise attenuation level γ. By following Gao's modified algorithm and with the help of results of [13], we can cast it into a nonlinear minimization problem. Although it is still probable that the global optimal solution is not reachable for γ, it is much easier to solve this problem by this method than the original non-convex problem.

IV. SIMULATIONS

Here we provide 2 examples regarding the H∞ controller design to demonstrate the effectiveness of the proposed method.

Example 1: In order to illustrate Theorem 1, we consider an unstable time-delay system with state-space equation (5) where

$$A=\begin{bmatrix}0&1&0\\0&0&1\\1&10&-8\end{bmatrix}, B=\begin{bmatrix}0\\0\\1\end{bmatrix}, C=\begin{bmatrix}1\\0\\0\end{bmatrix}^T, D_1=\begin{bmatrix}0\\0\\0.001\end{bmatrix}, D_2=\begin{bmatrix}0\\0\\0\end{bmatrix}, E=\begin{bmatrix}0.1\\0\\0\end{bmatrix} \quad (32)$$

Now, we consider the case that the size of the maximum delay is known and it is equal to 0.4sec. We apply Theorem 1 to find an H∞ composite state-derivative controller for the input-delayed system with state space matrices given in (32). Using iteration algorithm introduced in Remark 4, the minimum value for γ is obtained as 0.48. Table 1 shows the details of this result. The number of iterations in Table 1 denotes after how many iterations the stopping criterion, i.e. the conditions (7) ~ (9), was activated. The H∞ composite state-derivative controller with $\bar{\tau}=0.4\text{sec.}$ and γ=0.48 is given by

$$u(t)=[-3.76\ \ -17.92\ \ -0.4]x(t)+[-0.0013\ \ 0.007\ \ 0.0003]\dot{x}(t)$$

TABLE 1. CALCULATION RESULT TO OBTAIN SUBOPTIMAL MINIMUM γ

γ	Iterations
1	74
0.6	174
0.55	244
0.48	763

Remark 5: The iteration algorithm, mentioned in Remark 4, works efficiently for this example and many other examples. Nevertheless, it is still impossible to find an optimal solution for all the examples despite the existence of a solution. One way to deal with this problem is to solve an optimization problem similar to the one given in Remark 3 iteratively with BMI condition obtained during in the proof of Theorem 1. This condition is provided just before doing the congruence transformation. The disadvantage of this method is its dependence to the values of the initial conditions. In a general case, setting an appropriate initial condition is still an open problem.

Example 2: In this example, we apply the proposed approach to design a delay-dependent H∞ controller with composite state derivative feedback. The system under study is an active suspension system with a quarter-car model and actuator time-delay introduced in [5]. The state space equations are represented by the following equations

$$\begin{bmatrix}\dot{x}_1(t)\\\dot{x}_2(t)\\\dot{x}_3(t)\\\dot{x}_4(t)\end{bmatrix}=\begin{bmatrix}0&0&1&-1\\0&0&0&1\\k_s/m_s&0&-c_s/m_s&c_s/m_s\\k_s/m_u&-k_t/m_u&c_s/m_u&-(c_s+c_t)/m_u\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\\x_3(t)\\x_4(t)\end{bmatrix}$$
$$[0\ \ 0\ \ 1/m_s\ \ -1/m_u]^T u(t-\tau)+[0\ \ -1\ \ 0\ \ c_t/m_u]^T \dot{Z}_r(t)$$

Where m_s is the sprung mass and m_u is unsprung mass; k_s and k_t stands for suspension and tire stiffness, respectively; k_t and c_t are suspension and tire damping, respectively; Z_r is the road displacement input; Z_s and Z_u are the vertical displacement of the mass m_s and m_u, respectively; $u(t)$ is the control force usually provided by a hydraulic actuator; τ is the control input time delay. Moreover, $x_1(t)=Z_s-Z_u$ and $x_2(t)=Z_u-Z_r$ denote suspension travel and tire deflection, respectively; $x_3(t)$ is the sprung mass velocity and $x_4(t)$ denotes the unsprung mass velocity.

In order to have a good compromise between the different performance objectives, the controlled output is composed of Z_s-Z_u, Z_u-Z_r and $\ddot{Z}_u$. Therefore the vehicle suspension system is represented by the equation (6) where

$$C=\begin{bmatrix}-k_s/m_s&0&-c_s/m_s&c_s/m_s\\\alpha&0&0&0\\0&\beta&0&0\end{bmatrix}, D_1=\begin{bmatrix}1/m_s\\0\\0\end{bmatrix}, D_2=\begin{bmatrix}0\\0\\0\end{bmatrix}, w(t)=\dot{Z}_r(t)$$

where α and β are the positive scalar weightings for the suspension travel and tire deflection, respectively. These two parameters have been chosen as α=21 and β=42 in [5]. Consider m_s=972.2kg, m_u=113.6kg, k_s=42719.6N/m, k_t=101115N/m, c_s=1095Ns/m, c_t=14.6 and further assume -0.1m<Z_s - Z_u<0.1m. Before designing our proposed controller, we investigate the state feedback controller gain provided in [10] which is represented as

$$K=10^4\times[-0.3292\ \ -0.6361\ \ -1.0125\ \ -0.0020]$$

This controller stabilizes the system (6) with the H∞ performance index γ=11 and 0≤τ≤26ms. For sake of brevity, we denote this controller as controller I. In order to illustrate the effectiveness of our method, we design an H∞ composite state derivative feedback controller for the system under study. Considering the bandwidth requirement for disturbance rejection in human sensitivity range 0-65 rad/sec, a sensitivity weighting function is selected for the transfer function from $w(t)$ to $\ddot{Z}_s$ as $W(s)$=70/(s+70). Furthermore, we set α=1 and β=1 as the scalar weightings for the suspension travel and tire deflection and $\bar{\tau} = 40$ ms. Considering Remark 5, we obtain the following composite state-derivative feedback controller and denote it as controller II:

$$K_1 = 10^4 \times [3.3674 \quad 2.9964 \quad -1.2111 \quad 0.0148]$$

and $K_2 = [41.7275 \quad 37.016 \quad -13.8055 \quad 0.1737]$

with γ=5.2. To evaluate the performance of the active vehicle suspension, we investigate the transfer functions from $w(t)$ to $\ddot{Z}_s$ and $w(t)$ to Z_u-Z_r in the frequency range as shown in figs. 1-2.

It is observed from fig. 1. that applying the controller II in the closed loop system causes significant reduction in the magnitude of the transfer function from $w(t)$ to $\ddot{Z}_s$ compared to the controller I in human sensitivity range. Therefore, a better ride comfort is achieved for all constant time-delay $0<\tau\leq40$ *ms* in the desired frequency range. Fig. 2. illustrates the transfer function from $w(t)$ to Z_u-Z_r for both controllers I and II in the frequency range. As it is seen, applying controller II results less tire deflection in the frequencies 0-15 rad/sec and >50 rad/sec. in the compromise between the different performance objectives, a larger tire deflection is observed in frequencies 15 ~ 50 rad/sec compared to the controller I.

For a road disturbance input with 5cm height, the suspension travel of the closed-loop system with controller II is shown in fig. 3. in the frequency range. As it is seen in the fig. 3., suspension travel constraint is satisfied over the frequency range, whereas this criteria in passive system exceeds its limit in some frequencies.

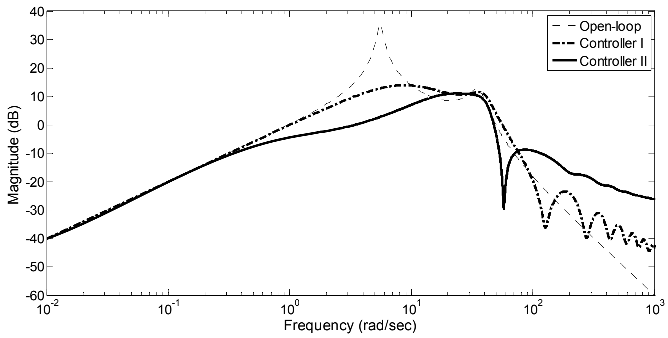

Fig. 1. Transfer function from $w(t)$ to $\ddot{Z}_s$ in the frequency range

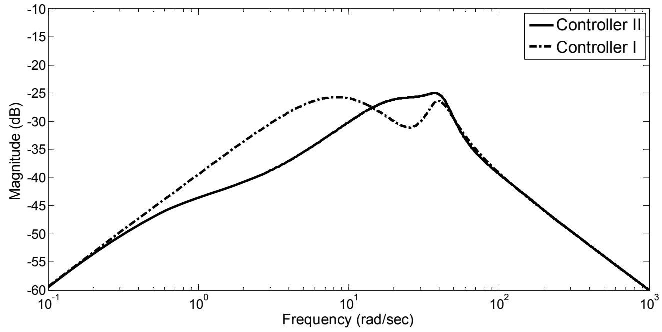

Fig. 2. Transfer function from $w(t)$ to Z_u-Z_r in the frequency range

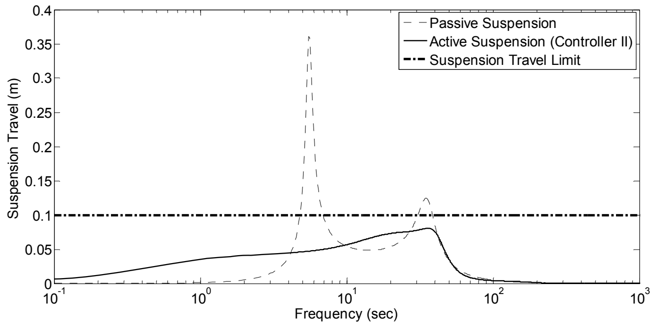

Fig. 3. Transfer function from $w(t)$ to Z_s-Z_u in the frequency range

V. Conclusions

H$_\infty$ control of a time-delay system with input delay is elaborated in this paper. The resulting closed-loop system with the proposed composite control law is a particular system of neutral type. In this system, the coefficients of delayed terms depend on the control law parameters. Since state-derivative feedback is a good remedy in practice, the composite control law is of great practical significance as well as theoretical importance. The Lyapunov theory is used to derive a set of delay-dependent sufficient conditions for the existence of an H$_\infty$ controller for the closed loop system. Moreover, two examples are presented in this paper to illustrate the effectiveness of our method. Numerical results show improvement in H$_\infty$ performance over the desired frequency range compared to the previous work.

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