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Delay-Dependent H_∞ Control of Linear Systems with Time-Varying Delays Using Proportional-Derivative State Feedback

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Abstract— **This paper considers H_∞ control problem for input-delayed systems for time-varying delays. A proportional-derivative state feedback control law is used in this paper. By this means, the resulting closed-loop system turns into a specific time-delay system of neutral type. The significant specification of this neutral system is that its delayed term coefficients depend on the controller parameters. This condition provides new challenging issues in theoretical research as well as providing new applications. In the present paper, new delay-dependent sufficient condition is derived for the existence of H_∞ controller in terms of matrix inequalities, in presence of varying time-delays. The resulting H_∞ controller guarantees asymptotic stability of the closed-loop system as well as a guaranteed limited system induced norm smaller than a prescribed level. Numerical examples are presented to illustrate the effectiveness of the proposed method.**

I. Introduction

Time-delay phenomena appear in many systems and processes, such as chemical and thermal processes [1], population dynamic model [2], rolling mill [3] and systems with long transmission line [4]. In many systems, time-delay is a source of instability. Hence, many researchers have paid great attention to the control of time-delay systems of retarded or neutral type.

Referring to H_∞ control of neutral systems, robust H_∞ state feedback control of uncertain neutral system has been considered in [5]. An optimization problem has been formulated with linear matrix inequality constraints to obtain an H_∞ state feedback controller. Observer-based H_∞ state feedback control for a class of uncertain neutral systems is another topic which Lien has considered in [6]. H_∞ output feedback control of neutral systems has also been the centre of attention in some research such as [7] and [8]. Moreover, Xu et al. have used bounded real lemma to design an H_∞ state feedback and positive real control for a linear neutral delay system [9].

A general representation of a neutral system is shown as

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{n} A_{h_i} x(t-h_i) + \sum_{j=1}^{k} A_{d_j} \dot{x}(t-d_j) + Bu(t) \qquad (1)$$

To the best knowledge of the authors, in all developed theories for neutral systems, no synthesis has been derived when both $x(t-h_i)$ and $\dot{x}(t-h_i)$ s' coefficients are dependent on the controller's parameter, whereas this condition has its own merits in practical application as well as leading to a new challenging theoretical problem. The importance of the above condition is observed in the control systems such as active vibration suppression. Du and Zhang proposed an H_∞ state-feedback controller for an input-delay active suspension system [10]. Since the ride comfort is an important objective which is related to the body acceleration sensed by the passenger, an acceleration feedback can effectively improve this performance objective in active suspension system or generally, in active vibration suppression systems. Some researchers have paid considerable attention to this idea such as [11] and [12]. Abdelaziz and válašek [12] proposed a formula similar to Ackermann for solving the pole-placement problem for non-delay linear single-input/single-output systems and multi-input/multi-output systems using state-derivative feedback. Assoncao et.al. [11] used this idea to design a stabilizing state-derivative controller for a delay-free system which bounds the output peak as well as the state-derivative feedback. Moreover, an analysis for the stability of a system controlled by proportional-derivative state feedback in presence of small uncertain delays in the feedback loop was presented in [17]. To the best of our knowledge, no synthesis of state-derivative feedback has been presented for input-delay systems in the literature, whereas, as described earlier, it could be of great significance in practice. To benefit the advantages of the state feedback as well as acceleration feedback or generally, state derivative feedback, we employ proportional-derivative state feedback control law as it is shown by the following equation:

$$u = K_1 x + K_2 \dot{x} \qquad (2)$$

Assume the general representation of linear input-delayed systems as follows:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{n} B_i u(t-h_i) + Ew(t) \qquad (3)$$

Applying the control law (2) to the input time-delay system (3) leads to a time-delay closed system of neutral type which is represented as follows:

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{n} B_i K_1 x(t-h_i) + \sum_{i=1}^{n} B_i K_2 \dot{x}(t-h_i) + Ew(t) \qquad (4)$$

As it is seen in the Equation (4), both $x(t\text{-}h_i)$ and $\dot{x}(t-h_i)$ s' coefficients are functions of the control law parameters. Therefore, finding K_1 and K_2 introduces a new challenging problem theoretically, whereas the choice of K_1 and K_2 can be very effective in obtaining desired performance in such applications. The main purpose of this paper is to elaborate this problem in detail and to design H_∞-based controller for the closed-loop system in presence of varying time-delay.

This paper is organized as follows. Problem formulation is introduced in Section 2, and in Section 3, an H_∞ controller is designed for the time-varying delay case. This is accomplished in terms of some matrix inequalities for the closed-loop time-delay system of neutral type. Illustrative examples are provided in section 4 to show the effectiveness of the proposed method in some case studies, and real application. Finally, the concluding remarks are given in section 5.

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$$\begin{bmatrix}
\Omega & BV-N_1-N_2 & BW-\bar{\tau}N_2 & 0 & 0 & 0 & E+LC^TD_2 & 0 & \bar{\tau}LA^T & LA^T & \bar{\tau}^2LA^TA^T & LA^TA^T & 0 & 0 & LC^T \\
* & -(1-\bar{\tau}_d)T & 0 & 0 & 0 & 0 & (D_1V)^TD_2 & 0 & \bar{\tau}(BV)^T & (BV)^T & \bar{\tau}^2(ABV)^T & (ABV)^T & 0 & 0 & (D_1V)^T \\
* & * & -\Theta_1-\bar{\tau}N_2 & 0 & 0 & 0 & (D_1W)^TD_2 & 0 & \bar{\tau}(BW)^T & (BW)^T & \bar{\tau}^2(ABW)^T & (ABW)^T & \bar{\tau}^2(BV)^T & (BV)^T & (D_1W)^T \\
* & * & * & -\Theta_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\tau}^2(BW)^T & (BW)^T & 0 \\
* & * & * & * & -\Theta_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -\Theta_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & D_2^TD_2-\gamma^2I & 0 & \bar{\tau}E^T & E^T & \bar{\tau}^2E^TA^T & E^TA^T & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\gamma^2I & 0 & 0 & \bar{\tau}^2E^T & E^T & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -\bar{\tau}F_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -H_1 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -\bar{\tau}^2F_2 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -H_2 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & -\bar{\tau}^2(\alpha\beta)^{-1}F_2 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & -\beta^{-1}H_2 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & * & * & * & -I
\end{bmatrix} < 0 \quad (7)$$

II. PROBLEM FORMULATION

In this paper, we consider the following time-delay system with input delay:

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t-\tau(t)) + Ew(t) \\
z(t) &= Cx(t) + D_1u(t-\tau(t)) + D_2w(t)
\end{aligned} \quad (5)$$

where x is the state, $w \in \Re^p$ is the disturbance input of system that belongs to $L_2[0,\infty)$, $\tau(t)$ is the time-varying delay of the system and is assumed to satisfy $0 < \tau(t) \le \bar{\tau}$ and $|\dot{\tau}(t)| < \bar{\tau}_d < 1/4$, $u \in \Re^m$ is the system input and $z \in \Re^q$ is the controlled system output. The matrices $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $E \in \Re^{n\times p}$, $C \in \Re^{q\times n}$, $D_1 \in \Re^{q\times m}$, $D_2 \in \Re^{q\times p}$ are assumed to be known. In this paper we assume that all the state variables are measured. Considering the control law (2), the state space equations of the closed-loop system is given by

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + BK_1x(t-\tau(t)) + BK_2\dot{x}(t-\tau(t)) + Ew(t) \\
z(t) &= Cx(t) + D_1K_1x(t-\tau(t)) + D_1K_2\dot{x}(t-\tau(t)) + D_2w(t) \\
x(t_0+\theta) &= \phi(\theta) \qquad \forall \theta \in [-\tau, 0]
\end{aligned} \quad (6)$$

Therefore, the resulting closed-loop system (6) is a time-delay system of neutral type which both coefficients of $x(t-\tau)$ and $\dot{x}(t-\tau)$ depending on the controller parameters. Here, we state the following lemma which will be used further in the main result of the paper.

Lemma 1 [14]: For a prescribed matrix, $M = \begin{bmatrix} A \\ B \end{bmatrix}$ or $M = \begin{bmatrix} A & B \end{bmatrix}$ we have the following inequality:

$$\max\{\bar{\sigma}(A), \bar{\sigma}(B)\} \le \bar{\sigma}(M) \le \sqrt{2}\max\{\bar{\sigma}(A), \bar{\sigma}(B)\}$$

III. H_∞ CONTROL DESIGN WITH TIME-VARYING DELAY

In many developed theories, conventional state feedback controller has been used for obtaining stability as well as performance objectives of the closed-loop system. In spite of the effectiveness of state feedback controller in many applications, it is not suitable for the cases that we need to have an acceleration feedback or generally derivative of the state in the feedback. On the other hand, H_∞ control is an effective method which guarantees asymptotic stability as well as performance objectives. This is why H_∞ control for time-delay systems has been among the most challenging topics in recent years. All the aforementioned facts motivate us to elaborate on the following Theorem and one Lemma which are stated in this section.

Theorem 1: Given scalars $\bar{\tau}, \bar{\tau}_d > 0$, the closed-loop system (6) is asymptotically stable and $\|T_{zw}\|_\infty < \gamma$, if there exist positive definite symmetric matrices L, T, H_1, H_2, F_1, F_2 $\in \Re^{n\times n}$, negative definite symmetric matrix N_2 and matrices M_1, M_2, N_1, $\in \Re^{n\times n}$, V, $W \in \Re^{m\times n}$ satisfying matrix inequalities (7) ~ (9). Moreover, H_∞ proportional-derivative state feedback control law is given by $u = VL^{-1}x + WL^{-1}\dot{x}$.

$$\begin{bmatrix} M_1 & N_1 \\ N_1^T & LF_1^{-1}L/2 \end{bmatrix} > 0 \quad (8)$$

$$\begin{bmatrix} 2M_2 & \bar{\tau}N_2 \\ \bar{\tau}N_2^T & \bar{\tau}(2+|\bar{\tau}_d|)^{-1}LF_2^{-1}L \end{bmatrix} > 0 \quad (9)$$

and,

$$\begin{bmatrix} L^TL & (BV)^T \\ BV & I \end{bmatrix} > 0 \quad (10)$$

in which $\Theta_1 = (1-\bar{\tau}_d)LH_1^{-1}L$, $\Theta_2 = (1-\bar{\tau}_d)(2+\bar{\tau}_d)^{-1}LH_2^{-1}L$, $\Theta_3 = (1-2\bar{\tau}_d)F_1/(2\bar{\tau}), \Theta_4 = (1-4\bar{\tau}_d)(2+\bar{\tau}_d)^{-1}F_2/(2\bar{\tau}^2)$, $\beta = 1+\bar{\tau}_d$

$$\Omega = LA^T + AL + N_1 + N_1^T + \bar{\tau}(M_1+M_2) + (2-\bar{\tau})N_2 + T$$

First, let us prove following useful lemma which will be applied in the proof of Theorem 1.

Lemma 2: Consider the neutral system (6) and assume $d(t) = [\,w^T(t) \quad \dot{w}^T(t)\,]^T$. If $\|T_{zd}\|_\infty < \gamma$ then the inequality $\|T_{zw}\|_\infty < \gamma$ is satisfied.

Proof: Since $z(s) = T_{zw}(s)w(s)$ and $d(s) = \begin{bmatrix} w(s) \\ sw(s) \end{bmatrix}$, then we have $z(s) = \begin{bmatrix} mT_{zw}(s) & \frac{1-m}{s}T_{zw}(s) \end{bmatrix} d(s)$.

where m is a real scalar value. Therefore $T_{zd}(s)$ can be written as

$$T_{zd}(s) = \begin{bmatrix} mT_{zw}(s) & \frac{1-m}{s}T_{zw}(s) \end{bmatrix}$$

By the Lemma 1 the following inequality holds as

$$\max\left\{\left\|mT_{zw}\right\|_\infty\ ,\ \left\|\frac{1-m}{s}T_{zw}\right\|_\infty\right\}\le\left\|T_{zd}(s)\right\|_\infty$$

Setting m=1, it can be easily concluded that if $\left\|T_{zd}(s)\right\|_\infty<\gamma$, then $\left\|T_{zw}\right\|_\infty<\gamma$. This completes the proof. ■

Corollary 1: Consider the neutral system (6) and two following performance indices:

$$J_1(w)=\int_0^\infty\left(z^Tz-\gamma^2w^Tw\right)d\tau,\ J_2(w)=\int_0^\infty(z^Tz-\gamma^2d^Td)d\tau$$

Where $d(t)$=[$w^T(t)\ \ \dot{w}^T(t)$]$^{\mathrm{T}}$. Since the inequalities J_1<0 and J_2<0 corresponds to H_∞ constraints $\left\|T_{zw}\right\|_\infty<\gamma$ and $\left\|T_{zd}\right\|_\infty<\gamma$ respectively, then for the inequality J_1<0 to be satisfied, it suffices to show that the condition J_2<0 is satisfied.

Proof of Theorem 1: In this case a Lyapunov-Krasovskii functional candidate for the system (6) has the form
$V=V_1+V_2+V_3+V_4$

Where

$$V_1=x(t)^TPx(t)\tag{11}$$

$$V_2=2\int_{-\tau(t)}^0\int_{t+\beta}^t\dot{x}^T(\alpha)Z_1\dot{x}(\alpha)d\alpha d\beta,\tag{12}$$

$$V_3=\int_{t-\tau(t)}^t x^T(\alpha)Qx(\alpha)d\alpha+\int_{t-\tau(t)}^t\dot{x}^T(\alpha)R_1\dot{x}(\alpha)d\alpha\tag{13}$$

$$\begin{aligned}V_4=&\int_{-\tau(t)}^0\int_{-\tau(t)}^\beta\int_{t+\eta}^t\ddot{x}^T(\alpha)Z'\ddot{x}(\alpha)d\alpha d\eta d\beta\\&+(1/2)\int_{-\tau(t)}^0\int_{-\tau(t)}^0\int_{t+\eta}^t\ddot{x}^T(\alpha)Z'\ddot{x}(\alpha)d\alpha d\eta d\beta\\&+(1/2)\int_{t-\tau(t)}^t\ddot{x}^T(\alpha)R'\ddot{x}(\alpha)d\alpha,\end{aligned}\tag{14}$$

where $P=P^{\mathrm{T}}>0$, $Q=Q^{\mathrm{T}}>0$, $R_1=R_1^{\mathrm{T}}>0$, $R'=R'^{\mathrm{T}}>0$, $Z_1=Z_1^{\mathrm{T}}>0$ and $Z'=Z'^{\mathrm{T}}>0$. Differentiating V_1 with respect to t gives us

$$\begin{aligned}\dot{V}_1=&2x^T(t)P\dot{x}(t)=2x^T(t)P\{Ax(t)+BK_1x(t-\tau(t))\\&+BK_2\dot{x}(t-\tau(t))+Ew(t)\}\end{aligned}$$

It is possible to write

$$x(t-\tau(t))=x(t)-\int_{t-\tau(t)}^t\dot{x}(\alpha)d\alpha\tag{15}$$

We introduce the following relation for the delayed derivative of the state:

$$\dot{x}(t-\tau)=\tau(t)^{-1}\left[x(t)-x(t-\tau(t))-\int_{-\tau(t)}^0\int_{t-\tau(t)}^{t+\beta}\ddot{x}(\alpha)d\alpha d\beta\right]\tag{16}$$

Therefore,

$$\begin{aligned}\dot{V}_1=&2x^TP(A+BK_1+\tau^{-1}BK_2)x(t)\\&-2\tau^{-1}x^TPBK_2x(t-\tau(t))-2x^TPBK_1\int_{t-\tau(t)}^t\dot{x}(\alpha)d\alpha\\&-2\tau^{-1}x^TPBK_2\int_{-\tau(t)}^0\int_{t-\tau(t)}^{t+\beta}\ddot{x}(\alpha)d\alpha d\beta+2x^T(t)PEw(t)\end{aligned}$$

Applying an extension of the proposed inequality in [13], the following upper bound for $\dot{V}_1$ is obtained:

$$\begin{aligned}&\dot{V}_1\le2x^TP\left(A+BK_1+\tau^{-1}BK_2\right)x(t)-2\tau^{-1}x^TPBK_2x(t-\tau(t))+\\&2x^T(t)PEw(t)+\int_{t-\tau(t)}^t\begin{bmatrix}x(t)\\\dot{x}(\alpha)\end{bmatrix}^T\begin{bmatrix}X_1&Y_1-PBK_1\\(Y-PBK_1)^T&Z_1\end{bmatrix}\begin{bmatrix}x(t)\\\dot{x}(\alpha)\end{bmatrix}d\alpha\\&+\tau^{-1}\int_{-\tau(t)}^0\int_{t-\tau(t)}^{t+\beta}\begin{bmatrix}x(t)\\\ddot{x}(\alpha)\end{bmatrix}^T\begin{bmatrix}X'&Y'-PBK_2\\(Y'-PBK_2)^T&\tau Z'\end{bmatrix}\begin{bmatrix}x(t)\\\ddot{x}(\alpha)\end{bmatrix}d\alpha d\beta\\&=x^T\{A^TP+PA+\tau\left(X_1+X'/2\right)+Y_1+Y_1^T\\&+\tau^{-1}\left(Y'+Y'^T\right)\}x+2x^T(t)\left(PBK_1-Y_1\right)x(t-\tau(t))\\&+2x^T(t)\left(PBK_2-Y'\right)\dot{x}(t-\tau(t))-2\tau^{-1}x^T(t)Y'x(t-\tau(t))\\&+2x^T(t)PEw(t)+\int_{t-\tau(t)}^t\dot{x}^T(\alpha)Z_1\dot{x}(\alpha)d\alpha\\&+\int_{-\tau(t)}^0\int_{t-\tau(t)}^{t+\beta}\ddot{x}^T(\alpha)Z'\ddot{x}(\alpha)d\alpha d\beta\end{aligned}\tag{17}$$

where

$$\begin{bmatrix}X_1&Y_1\\Y_1^T&Z_1\end{bmatrix}>0\tag{18}$$

and,

$$\begin{bmatrix}X'&Y'\\Y'^T&\tau Z'\end{bmatrix}>0\tag{19}$$

Also, the time derivative of V_2 can be represented as follows:

$$\begin{aligned}\dot{V}_2=&2\tau(t)\dot{x}^T(t)Z_1\dot{x}(t)-2\int_{t-\tau(t)}^t\dot{x}^T(\alpha)Z_1\dot{x}(\alpha)d\alpha\\&+2\dot{\tau}\int_{t-\tau(t)}^t\dot{x}^T(\alpha)Z_1\dot{x}(\alpha)d\alpha\\&=2\tau(t)\dot{x}^T(t)Z_1\dot{x}(t)-\int_{t-\tau(t)}^t\dot{x}^T(\alpha)Z_1\dot{x}(\alpha)d\alpha\\&-(1-2\dot{\tau})\int_{t-\tau(t)}^t\dot{x}^T(\alpha)Z_1\dot{x}(\alpha)d\alpha\end{aligned}$$

Using Lemma 1 in [18], we have

$$\begin{aligned}\dot{V}_2\le&\left(Ax(t)+BK_1x(t-\tau)+BK_2\dot{x}(t-\tau)+Ew(t)\right)^T.\\&(2\tau Z_1)\left(Ax(t)+BK_1x(t-\tau)+BK_2\dot{x}(t-\tau)+Ew(t)\right)\\&-\int_{t-\tau}^t\dot{x}^T(\alpha)Z_1\dot{x}(\alpha)d\alpha\\&-\left(\frac{1-2\dot{\tau}}{\tau}\right)\left(\int_{t-\tau(t)}^t\dot{x}^T(\alpha)d\alpha\right)Z_1\left(\int_{t-\tau(t)}^t\dot{x}(\alpha)d\alpha\right)\end{aligned}\tag{20}$$

It can be shown that the time derivative of V_3 and V_4 are

$$\begin{aligned}\dot{V}_3=&x^T(t)Qx(t)-(1-\dot{\tau})x^T(t-\tau)Qx(t-\tau)\\&+\dot{x}^T(t)R_1\dot{x}(t)-(1-\dot{\tau})\dot{x}^T(t-\tau)R_1\dot{x}(t-\tau)\end{aligned}\tag{21}$$

$$\begin{aligned}\dot{V}_4=&\ddot{x}^T(t)\left(\tau^2Z'+R'/2\right)\ddot{x}(t)-\left(\frac{1-\dot{\tau}}{2}\right)\ddot{x}^T(t-\tau(t))R'\ddot{x}(t-\tau(t))\\&-\int_{-\tau(t)}^0\int_{t-\tau(t)}^{t+\beta}\ddot{x}^T(\alpha)Z'\ddot{x}(\alpha)d\alpha d\beta\\&-(1/2)\int_{-\tau(t)}^0\int_{t-\tau(t)}^t\ddot{x}^T(\alpha)Z'\ddot{x}(\alpha)d\alpha d\beta\\&+(3\dot{\tau}/2)\int_{-\tau(t)}^0\int_{t-\tau(t)}^t\ddot{x}^T(\alpha)Z'\ddot{x}(\alpha)d\alpha d\beta\\&+(\dot{\tau}/2)\int_{-\tau(t)}^0\int_{t+\beta}^t\ddot{x}^T(\alpha)Z'\ddot{x}(\alpha)d\alpha d\beta\\&\le\ddot{x}^T(t)\left(\tau^2Z'+R'/2\right)\ddot{x}(t)-\left(\frac{1-\dot{\tau}}{2}\right)\ddot{x}^T(t-\tau(t))R'\ddot{x}(t-\tau(t))\end{aligned}$$

$$-\int_{-\tau(t)}^{0}\int_{t-\tau(t)}^{t+\beta}\ddot{x}^T(\alpha)Z'\ddot{x}(\alpha)d\alpha d\beta$$
$$-\left(\frac{1-4\dot{\tau}}{2}\right)\int_{-\tau(t)}^{0}\int_{t-\tau(t)}^{t}\ddot{x}^T(\alpha)Z'\ddot{x}(\alpha)d\alpha d\beta$$
$$\le \ddot{x}^T(t)\left(\tau^2 Z'+R'/2\right)\ddot{x}(t)-\left(\frac{1-\dot{\tau}}{2}\right)\ddot{x}^T(t-\tau(t))R'\ddot{x}(t-\tau(t))$$
$$-\int_{-\tau(t)}^{0}\int_{t-\tau(t)}^{t+\beta}\ddot{x}^T(\alpha)Z'\ddot{x}(\alpha)d\alpha d\beta$$
$$-\left(\frac{1-4\dot{\tau}}{2\tau^2}\right)\left(\int_{-\tau(t)}^{0}\int_{t-\tau(t)}^{t}\ddot{x}^T(\alpha)d\alpha d\beta\right)Z'\left(\int_{-\tau(t)}^{0}\int_{t-\tau(t)}^{t}\ddot{x}(\alpha)d\alpha d\beta\right) \tag{22}$$

Therefore we have

$$\dot{V}(t)=\sum_{i=1}^{4}\dot{V}_i \tag{23}$$

We consider (17), (20) ~ (23) and define $\tau^{-1}Y'=Y_2$. By the assumption of $Y'=Y'^T<0$ and then adding and subtracting the terms $x^T(t)(\tau Y_2)x(t)$ and $\dot{x}^T(t-\tau)(\tau Y_2)^T\dot{x}(t-\tau)$ in (23), an upper bound for $\dot{V}$ is obtained as

$$\begin{aligned}\dot{V}(t)&=\sum_{i=1}^{4}\dot{V}_i\le x^T\{A^TP+PA+\bar{\tau}\left(X_1+X'/2\right)+Y_1+Y_1^T\\
&+\left(Y_2+Y_2^T\right)\}x+2x^T(t)\left(PBK_1-Y_1\right)x(t-\tau(t))\\
&+2x^T(t)PBK_2\dot{x}(t-\tau(t))+\left(x^T(t)+\dot{x}(t-\tau(t))\right)^T(-\bar{\tau}Y_2).\\
&\left(x^T(t)+\dot{x}(t-\tau(t))\right)-x^T(t)Y_2x(t-\tau(t))-x^T(t-\tau(t))Y_2x(t)\\
&+\dot{x}^T(t)(2\bar{\tau}Z_1)\dot{x}(t)-\left(\frac{1-2\bar{\tau}_d}{\bar{\tau}}\right)\left(\int_{t-\tau(t)}^{t}\dot{x}^T(\alpha)d\alpha\right)Z_1\left(\int_{t-\tau(t)}^{t}\dot{x}(\alpha)d\alpha\right)\\
&+x^T(t)Qx(t)-(1-\bar{\tau}_d)x^T(t-\tau(t))Qx(t-\tau(t))+\dot{x}^T(t)R_1\dot{x}(t)\\
&-(1-\bar{\tau}_d)\dot{x}^T(t-\tau(t))R_1\dot{x}(t-\tau(t))+\ddot{x}^T(t)\left(\bar{\tau}^2Z'+R'/2\right)\ddot{x}(t)\\
&-\left(\frac{1-\bar{\tau}_d}{2}\right)\ddot{x}^T(t-\tau(t))R'\ddot{x}(t-\tau(t))\\
&-\left(\frac{1-4\bar{\tau}_d}{2\bar{\tau}^2}\right)\left(\int_{-\tau(t)}^{0}\int_{t-\tau(t)}^{t}\ddot{x}^T(\alpha)d\alpha d\beta\right)Z'\left(\int_{-\tau(t)}^{0}\int_{t-\tau(t)}^{t}\ddot{x}(\alpha)d\alpha d\beta\right)\\
&+x^T(t)(\tau Y_2)x(t)+\dot{x}^T(t-\tau)(\tau Y_2)^T\dot{x}(t-\tau)=\zeta^T\Pi\zeta+\\
&x^T(t)(\tau(t)Y_2)x(t)+\dot{x}^T(t-\tau)(\tau(t)Y_2)^T\dot{x}(t-\tau)\end{aligned} \tag{24}$$

Assume zero initial condition, i.e. $\phi(t)=0$, $\forall t\in[-\tau,0]$ we have $V(q(t))|_{t=0}=0$. For a prescribed $\gamma>0$, consider the following performance index J_2 in Corollary 1. Therefore, J_2 can be rewritten as

$$J_{zd}(w)=\int_0^{\infty}\left(z^Tz-\gamma^2w^Tw-\gamma^2\dot{w}^T\dot{w}\right)d\tau \tag{25}$$

Since $V(t)|_{t=0}=0$ and $V(t)|_{t\to\infty}\ge 0$, we obtain

$$\begin{aligned}J_{zd}(w)&=\int_0^{\infty}\left(z^Tz-\gamma^2w^Tw-\gamma^2\dot{w}^T\dot{w}+\dot{V}(t)\right)d\tau+V(t)\big|_{t=0}-V(t)\big|_{t\to\infty}\\
&\le\int_0^{\infty}\left(z^Tz-\gamma^2w^Tw-\gamma^2\dot{w}^T\dot{w}+\dot{V}(t)\right)d\tau\end{aligned}$$

Hence the following inequality is obtained:

$$\begin{aligned}J_{zd}(w)\le\int_0^{\infty}&\{x^TC^TCx+2x^TC^TD_1K_1x(t-\tau)+2x^TC^TD_1K_2\dot{x}(t-\tau)\\
&+2x^TC^TD_2w+x^T(t-\tau)K_1^TD_1^TD_1K_1x(t-\tau)+2x^T(t-\tau)K_1^TD_1^TD_2w\\
&+2x^T(t-\tau)K_1^TD_1^TD_1K_2\dot{x}(t-\tau)+\dot{x}^T(t-\tau)K_2^TD_1^TD_1K_2\dot{x}(t-\tau)\\
&+2\dot{x}^T(t-\tau)K_2^TD_1^TD_2w+w^TD_2^TD_2w-\gamma^2w^Tw-\gamma^2\dot{w}^T\dot{w}+\dot{V}(t)\}d\tau\end{aligned} \tag{26}$$

Considering (24), $0<\tau(t)\le\bar{\tau}$ and $|\dot{\tau}(t)|<\bar{\tau}_d<1/4$ a new upper bound for (26) is obtained as

$$J_{zd}\le \int_0^{\infty}\left\{\zeta^T\Pi\zeta+x^T(t)(\tau(t)Y_2)x(t)+\dot{x}^T(t-\tau)(\tau(t)Y_2)^T\dot{x}(t-\tau)\right\}d\tau$$

with defined

$$\zeta=\begin{bmatrix}x(t) & x(t-\tau) & \dot{x}(t-\tau) & \ddot{x}(t-\tau) & \zeta_1 & \zeta_2 & w(t) & \dot{w}(t)\end{bmatrix}$$

where $\zeta_1=\int_{t-\tau(t)}^{t}\dot{x}(\alpha)d\alpha$, $\zeta_2=\int_{-\tau(t)}^{0}\int_{t-\tau(t)}^{t}\ddot{x}(\alpha)d\alpha d\beta$

and $\Pi=\left[\Sigma_{ij}\right]$ where $\Sigma_{ij}=\Sigma_{ij}^T$ and i,j= 1,2,…8.

in which,

$$\begin{aligned}&\Sigma_{11}=A^TP+PA+Y_1+Y_1^T+\bar{\tau}\left(X_1+X'/2\right)+(2-\bar{\tau})Y_2\\
&\quad+A\Upsilon_1A+A^TA^T\Upsilon_2AA+Q+C^TC\\
&\Sigma_{12}=PBK_1-Y_1-Y_2+A^T\Upsilon_1BK_1+A^TA^T\Upsilon_2ABK_1+C^TD_1K_1\\
&\Sigma_{13}=PBK_2-\bar{\tau}Y_2+A^T\Upsilon_1BK_2+A^TA^T\Upsilon_2ABK_2+C^TD_1K_2\\
&\Sigma_{14}=0,\qquad \Sigma_{17}=PE+A^T\Upsilon_1E+A^TA^T\Upsilon_2AE+C^TD_2\\
&\Sigma_{18}=A^TA^T\Upsilon_2E\\
&\Sigma_{22}=-(1-\bar{\tau}_d)Q+(BK_1)^T\Upsilon_1BK_1+(ABK_1)^T\Upsilon_2ABK_1+K_1^TD_1^TD_1K_1\\
&\Sigma_{23}=(BK_1)^T\Upsilon_1BK_2+(ABK_1)^T\Upsilon_2(ABK_2)+K_1^TD_1^TD_1K_2\\
&\Sigma_{24}=0,\quad \Sigma_{27}=(BK_1)^T\Upsilon_1E+(ABK_1)^T\Upsilon_2AE+K_1^TD_1^TD_2\\
&\Sigma_{28}=(ABK_1)^T\Upsilon_2E\\
&\Sigma_{33}=-(1-\bar{\tau}_d)R_1-\bar{\tau}Y_2+(BK_2)^T\Upsilon_1BK_2\\
&\quad+(ABK_2)^T\Upsilon_2(ABK_2)+(1+\bar{\tau}_d)(BK_1)^T\Upsilon_2BK_1+K_2^TD_1^TD_1K_2\\
&\Sigma_{34}=(1+\bar{\tau}_d)(BK_1)^T\Upsilon_2BK_2,\\
&\Sigma_{37}=(BK_2)^T\Upsilon_1E+(ABK_2)^T\Upsilon_2AE+K_2^TD_1^TD_2,\ \Sigma_{38}=(ABK_2)^T\Upsilon_2E,\\
&\Sigma_{44}=-(1-\bar{\tau}_d)R'/2+(1+\bar{\tau}_d)(BK_2)^T\Upsilon_2BK_2\\
&\Sigma_{48}=0,\ \Sigma_{55}=-(1-2\bar{\tau}_d)Z_1/\bar{\tau},\ \Sigma_{66}=-(1-4\bar{\tau}_d)Z'/\left(2\bar{\tau}^2\right)\\
&\Sigma_{77}=E^T\Upsilon_1E+E^TA^T\Upsilon_2AE+D_2^TD_2-\gamma^2I,\ \Sigma_{78}=E^TA^T\Upsilon_2E,\\
&\Sigma_{88}=E^T\Upsilon_2E-\gamma^2I\\
&\Sigma_{15}=0,\ \Sigma_{16}=0,\ \Sigma_{25}=0,\ \Sigma_{26}=0,\ \Sigma_{35}=0,\ \Sigma_{36}=0,\ \Sigma_{45}=0\\
&\Sigma_{46}=0,\ \Sigma_{47}=0,\ \Sigma_{56}=0,\ \Sigma_{57}=0,\ \Sigma_{58}=0,\ \Sigma_{67}=0,\ \Sigma_{68}=0\end{aligned}$$

where $\Upsilon_1=R_1+2\bar{\tau}Z_1$ and $\Upsilon_2=(2+\bar{\tau}_d)\left(\bar{\tau}^2Z'+R'/2\right)$. (27)

Considering the constraint $Y_2=Y_2^T<0$, if $\zeta^T\Pi\zeta<0$, then the negative semi definiteness of J_{zd} in (26) is guaranteed for any varying time-delay $\tau(t)$ satisfying $0<\tau(t)\le\bar{\tau}$ and $|\dot{\tau}(t)|<\bar{\tau}_d<1/4$. Hence, when assuming $w(t),\dot{w}(t)\in L_2[0\ \ \infty)$ and $\Pi<0$ then implies that $J_{zd}<0$ and therefore $\|T_{zd}\|_\infty<\gamma$. This condition is the H_∞ performance to guarantee the tracking performance. By Lemma 2, the inequality $\|T_{zd}\|_\infty<\gamma$ guarantees $\|T_{zw}\|_\infty<\gamma$ to be satisfied. Using Schur complement, the condition $\Pi<0$ is equivalent to the following matrix inequality:

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} \\ \Xi_{12}^T & \Xi_{22} \end{bmatrix} < 0 \quad (28)$$

with LMI (18) and

$$\begin{bmatrix} X' & \tau Y_2 \\ \tau Y_2 & \tau Z' \end{bmatrix} > 0 \quad (29)$$

where

$$\Xi_{11} = \begin{bmatrix} \Omega_1 & PBK_1 - Y_1 - Y_2 & PBK_2 - \overline{\tau} Y_2 & 0 & PE + C^T D_2 & 0 \\ * & -(1-\overline{\tau}_d)Q & 0 & 0 & K_1^T D_1^T D_2 & 0 \\ * & * & -(1-\overline{\tau}_d)R_1 - \overline{\tau} Y_2 & 0 & K_2^T D_1^T D_2 & 0 \\ * & * & * & -\Gamma & 0 & 0 \\ * & * & * & * & D_2^T D_2 - \gamma^2 I & 0 \\ * & * & * & * & * & -\gamma^2 I \end{bmatrix}$$

with

$$\Gamma = diag\left((1-\overline{\tau}_d)R'/2,\ (1-2\overline{\tau}_d)Z_1/\overline{\tau},\ (1-4\overline{\tau}_d)Z'/(2\overline{\tau}^2)\right)$$

$$\Xi_{12} = \begin{bmatrix} \overline{\tau}\Delta_1 & \Delta_1 & \overline{\tau}^2\Delta_2 & \Delta_2 & \overline{\tau}^2\Delta_3 & \Delta_3 & \Delta_4 \end{bmatrix}$$

where

$$\Omega_1 = A^T P + PA + Y_1 + Y_1^T + \overline{\tau}\left(X_1 + (1/2)X'\right) + (2-\overline{\tau})Y_2 + Q$$

$$\Delta_1 = \begin{bmatrix} A & BK_1 & BK_2 & 0 & 0 & 0 & E & 0 \end{bmatrix}^T$$

$$\Delta_2 = \begin{bmatrix} AA & ABK_1 & ABK_2 & 0 & 0 & 0 & AE & E \end{bmatrix}^T$$

$$\Delta_3 = \begin{bmatrix} 0 & 0 & BK_1 & BK_2 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\Delta_4 = \begin{bmatrix} C & D_1K_1 & D_1K_2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

and

$$\Xi_{22} = -diag\left(\overline{\tau}Z_1^{-1}/2, R_1^{-1}, \overline{\tau}^2\alpha^{-1}Z'^{-1}, 2\alpha^{-1}R'^{-1}, \overline{\tau}^2(\alpha\beta)^{-1}Z'^{-1}, 2(\alpha\beta)^{-1}R'^{-1}, I\right)$$

where $\alpha = 2 + \overline{\tau}_d$ and $\beta = 1 + \overline{\tau}_d$. Denote $P^{-1}, Z_1^{-1}/2, \alpha^{-1}Z'^{-1}, (\alpha\beta)^{-1}Z'^{-1}, R_1^{-1}, 2\alpha^{-1}R'^{-1}, 2(\alpha\beta)^{-1}R'^{-1}$ as L, F_1, F_2, F_3, H_1, H_2 and H_3 respectively, by performing a congruence transformation to (28) by diag($L, L, L, L, F_1, F_2, I, I, I, I, I, I, I, I, I$) together with introducing the change of variables $M_1 = LX_1L$, $M_2 = L(X'/2)L$, $N_1 = LY_1L$, $N_2 = LY_2L$, $T = LQL$, $V = K_1L$, $W = K_2L$, the matrix inequality (7) is derived. Furthermore, pre and post multiplying the LMI (18) by diag (L, L) and its transpose and defining the same change of variables, the matrix inequality (8) is provided.

Similarly, by performing a congruence transformation to (29) by diag (L, L) and using Schur complement, we have

$$LX'L - (\tau LY_2L)(\tau LZ'L)^{-1}(\tau LY_2^T L) > 0$$

Substituting $M_2 = L(X'/2)L,\ N_2 = LY_2L$ and $F_2 = \alpha^{-1}Z'^{-1}$, the following matrix inequality is derived.

$$2M_2 - (\tau N_2)(\tau\alpha^{-1}LF_2^{-1}L)^{-1}(\tau N_2^T) > 0 \quad (30)$$

On the other hand we have

$$2M_2 - (\tau N_2)(\tau\alpha^{-1}LF_2^{-1}L)^{-1}(\tau N_2)^T \geq 2M_2 - (\overline{\tau}N_2)(\overline{\tau}\alpha^{-1}LF_2^{-1}L)^{-1}(\overline{\tau}N_2)^T \quad (31)$$

Therefore, satisfying the following inequality guarantees the inequality (30) to be satisfied.

$$2M_2 - (\overline{\tau}N_2)(\overline{\tau}\alpha^{-1}LF_2^{-1}L)^{-1}(\overline{\tau}N_2)^T > 0 \quad (32)$$

Applying Schur complement, the matrix inequality (9) is obtained. To guarantee asymptotic stability of the difference operator $\mathcal{D}(x_t) = x(t) - BK_2x(t-\tau)$, it suffices to guarantee $\overline{\sigma}(BK_2) < 1$ or $(BK_2)^T(BK_2) < I$. Using Schur complement and performing a congruence transformation by diag (L^T, I), the matrix inequality (10) is provided. This completes the proof.■

Remark 2: It should be noted that generally, the problem of finding the smallest $\gamma > 0$, namely γ_0, can be computed by solving the following optimization problem in L, T, H_1, H_2, F_1, $F_2 > 0$, $N_2 < 0$ and $\sigma = \gamma^2$:

Minimize σ

Subject to L, T, H_1, H_2, F_1, $F_2 > 0$, $N_2 < 0$, $\sigma > 0$ and matrix inequality conditions (7) ~ (9)

Remark 3: Note that, the resulting conditions presented in the Theorem 1 are not LMI conditions. Gao and Wang [15] presented a modified algorithm using Moon's idea to find a minimum noise attenuation level γ. By following Gao's modified algorithm and with the help of results of [16], we can cast it into a nonlinear minimization problem which is much to solve than the original non-convex problem.

IV. Simulations

Here we provide 3 examples regarding the H_∞ controller design to demonstrate the effectiveness of the proposed method.

Example 1: In order to illustrate Theorem 1, we consider an unstable time-delay system with state-space equation (5) where

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, C = I, D_1 = \begin{bmatrix} 0 \\ 0.001 \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, E = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \quad (33)$$

Now, we consider the case that $\overline{\tau} = 0.075\sec$ and $\overline{\tau}_d = 0.08$. We then apply Theorem 1 to find an H_∞ proportional-derivative state feedback controller for the input-delayed system with state space matrices given in (33). Using iteration algorithm introduced in Remark 3, the minimum value for γ is obtained as 0.32. Table 1 shows the details of this result. The number of iterations in Table 1 denotes after how many iterations the stopping criterion, i.e. the conditions (7) ~ (9), was activated. The H_∞ proportional-derivative state feedback controller with $\overline{\tau} = 0.075\sec$, $\overline{\tau}_d = 0.08$. and γ=1.06 is given by

$$u(t) = [19.3036 \quad -38.2908]x(t) + [-0.0036 \quad -0.0228]\dot{x}(t)$$

TABLE 1. CALCULATION RESULT TO OBTAIN SUBOPTIMAL MINIMUM γ

γ	Iterations
5	175
3	178
1.7	181
1.06	217

Remark 4: The iteration algorithm, mentioned in Remark 3, works efficiently for this example and many other examples. Nevertheless, it is still impossible to find an optimal solution for all the examples despite the fact that a solution exists. One way to deal with this problem is to solve an optimization problem similar to the one given in Remark 2 iteratively with BMI condition obtained in the proof of Theorem 1. This condition is provided just before doing the congruence transformation.

Example 2: Consider the vibration suppression of a platform which has been presented in [17]. A state derivative feedback, i.e. the control law (2) with K_1=0 and

$K_2 = \begin{bmatrix} 19.64 & 5.899 & 0.22 & -0.21 \\ 8.367 & 49.0 & -0.15 & 0.12 \end{bmatrix}$, was used to control the closed loop system. We denote this controller as controller A. This controller is applied to the closed-loop system using a first order filter introduced in [17]. The transient response from an initial state x(0)=[-0.01 0.02 -0.02 0.01]^T can be seen in fig (1). As it is shown in fig. (1), the stability of the closed-loop system is destroyed for $\bar{\tau} = 0.7\,\text{sec}$ due to high frequency oscillations. Now using Theorem 1 and setting $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, E = \begin{bmatrix} 0 & 0 & 0.1 & -0.1 \end{bmatrix}^T$,

we obtain a state-derivative feedback with K_1=0 and

$$K_2 = \begin{bmatrix} 1.516 & -0.5 & -0.46 & -0.436 \\ 0.452 & 1.172 & -0.4 & -0.452 \end{bmatrix}$$

which guarantees the stability of the closed loop system with $\bar{\tau} = 6\,\text{sec}, \bar{\tau}_d = 0.2$ and the performance index γ=0.038. We denoted this controller as controller B. The transient response of the feedback system in presence of the same delay $\bar{\tau} = 0.7\,\text{sec}$ is shown in fig. (2). As can be seen, our controller provides the stability of the closed-loop system with a fast and well damped response.

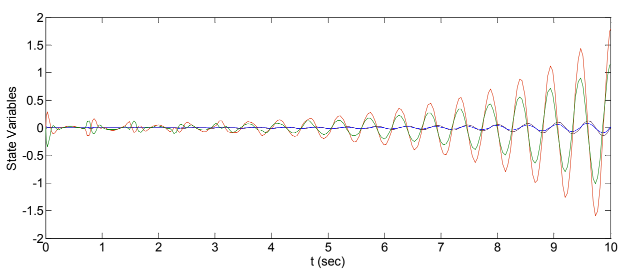

Fig. 1. Response of the feedback system of Example (2) with controller A

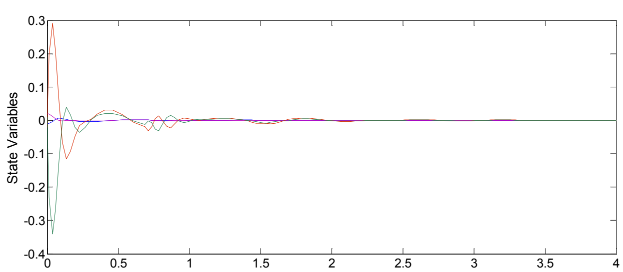

Fig. 2. Response of the feedback system of Example (2) with controller B

Example 3: In this example, we apply the proposed approach to design a delay-dependent H$_\infty$ controller with proportional-derivative state feedback. The system under study is an active suspension system with a quarter-car model and time-varying input delay introduced in [10]. The state space equations are represented by the following equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ k_s/m_s & 0 & -c_s/m_s & c_s/m_s \\ k_s/m_u & -k_t/m_u & c_s/m_u & -(c_s+c_t)/m_u \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$
$$+\begin{bmatrix} 0 & 0 & 1/m_s & -1/m_u \end{bmatrix}^T u(t-\tau) + \begin{bmatrix} 0 & -1 & 0 & c_t/m_u \end{bmatrix}^T \dot{Z}_r(t)$$

Where m_s is the sprung mass and m_u is unsprung mass; k_s and k_t stands for suspension and tire stiffness, respectively; k_t and c_t are suspension and tire damping, respectively; Z_r is the road displacement input; Z_s and Z_u are the vertical displacement of the mass m_s and m_u, respectively; $u(t)$ is the control force usually provided by a hydraulic actuator; τ is the control input time-delay. Moreover, $x_1(t)=Z_s-Z_u$ and $x_2(t)=Z_u-Z_r$ denote suspension travel and tire deflection, respectively; $x_3(t)$ is the sprung mass velocity and $x_4(t)$ denotes the unsprung mass velocity.

In order to have a good compromise between the different performance objectives, the controlled output is composed of Z_s-Z_u, Z_u-Z_r and $\ddot{Z}_u$. Therefore the vehicle suspension system is represented by the equation (6) where

$$C = \begin{bmatrix} -k_s/m_s & 0 & -c_s/m_s & c_s/m_s \\ \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix}, D_1 = \begin{bmatrix} 1/m_s \\ 0 \\ 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, w(t) = \dot{Z}_r(t)$$

where α and β are the positive scalar weightings for the suspension travel and tire deflection, respectively. These two parameters have been chosen as α =21 and β =42 in [10]. Consider m_s = 972.2kg, m_u = 113.6kg, k_s = 42719.6N/m, k_t = 101115N/m, c_s = 1095Ns/m, c_t = 14.6 and further assume -0.1m < Z_s - Z_u < 0.1m. Before designing our proposed controller, we investigate the state feedback controller gain provided in [10] which is represented as

$$K = 10^4 \times \begin{bmatrix} -0.3292 & -0.6361 & -1.0125 & -0.0020 \end{bmatrix}$$

This controller stabilizes the system (6) with the H$_\infty$ performance index γ=11 and a constant time-delay $0 \le \tau(t) \le$ 26ms. For sake of brevity, we denote this controller as controller I. In order to illustrate the effectiveness of our method, we design an H$_\infty$ proportional-derivative state feedback controller for the system under study. Considering the bandwidth requirement for disturbance rejection in human sensitivity range 0-65 rad/sec, a sensitivity weighting function is selected for the transfer function from $w(t)$ to $\ddot{Z}_s$ as $W(s)$=70/(s+70). Furthermore, we set α=21, β=42 (as considered in [10]), $\bar{\tau} = 40\,\text{ms}$ and $\bar{\tau}_d = 0.02$. Considering Remark 4, we obtain the following proportional-derivative state feedback controller and denote it as controller II:

$$K_1 = 10^4 \times \begin{bmatrix} 3.24 & 3.2 & -0.64 & 0.018 \end{bmatrix}$$

and, $$K_2 = \begin{bmatrix} 6.3 & 3.36 & -1.73 & 0.0048 \end{bmatrix}$$

with γ =8.28. To evaluate the performance of the active vehicle suspension, we investigate the transfer functions from $w(t)$ to $\ddot{Z}_s$ and $w(t)$ to Z_u-Z_r in the frequency domain as shown in figs. 3-4.

It is observed from fig. 3 that applying the controller II in the closed loop system causes significant reduction in the magnitude of the transfer function from $w(t)$ to $\ddot{Z}_s$ compared to the controller I in human sensitivity range. Therefore, a better ride comfort is achieved for all varying time-delay $0 \leq \tau \leq 40$ *ms* in the desired frequency range. Fig. 4. illustrates the transfer function from $w(t)$ to Z_u-Z_r for both controllers I and II in the frequency range. As it is seen, applying controller II results less tire deflection in the frequencies 0-15 rad/sec and >50 rad/sec. in the compromise between the different performance objectives, a larger tire deflection is observed in frequencies 15 ~ 50 rad/sec compared to the controller I.

For a road disturbance input with 5cm height, the suspension travel of the closed-loop system with controller II is shown in fig. 5. in the frequency range. As it is seen in the fig. 5., suspension travel constraint is satisfied over the frequency range, whereas this criteria in passive system exceeds its limit in some frequencies.

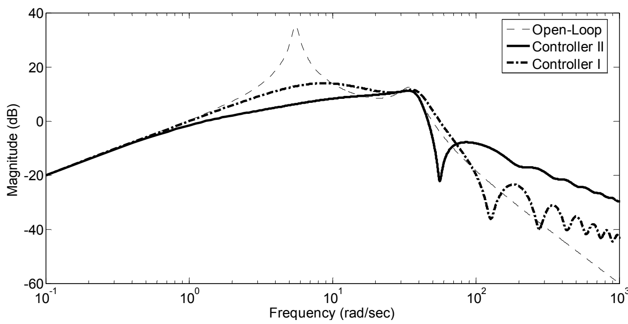

Fig. 3. Transfer function from $w(t)$ to $\ddot{Z}_s$ in the frequency range

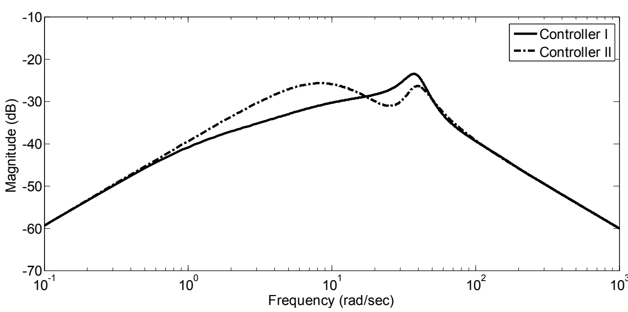

Fig. 4. Transfer function from $w(t)$ to Z_u-Z_r in the frequency range

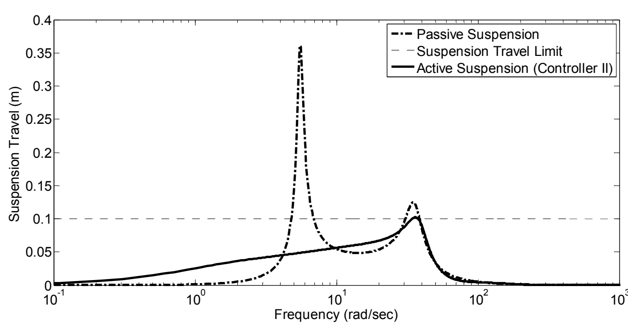

Fig. 5. Transfer function from $w(t)$ to Z_s-Z_u in the frequency range

V. CONCLUSIONS

H_∞ control of a time-delay system with input delay for varying time-delay case is elaborated in this paper. The resulting closed-loop system with the proposed control law is a particular system of neutral type. In this system, the coefficients of delayed terms depend on the control law parameters. Since state-derivative feedback is a good remedy in practice, the proposed dynamic control law is of great practical significance as well as theoretical importance. The Lyapunov theory is used to derive a set of delay-dependent sufficient conditions in presence of varying time-delay. A sufficient condition is derived for the existence of an H_∞ controller for the closed loop system in terms of matrix inequalities. Moreover, three examples are presented in this paper to illustrate the effectiveness of our method. Simulations show improvement in H_∞ performance over the desired frequency range compared to the previous work.

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