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Short Papers

An Analytic-Iterative Redundancy Resolution Scheme for Cable-Driven Redundant Parallel Manipulators

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***Abstract*—In this paper, redundancy resolution of a cable-driven parallel manipulator is performed through an analytic-iterative scheme. The redundancy resolution scheme is formulated as a convex optimization problem with inequality constraints that are imposed by manipulator structure and cable dynamics. The Karush–Kuhn–Tucker theorem is used to analyze the optimization problem and to draw an analytic-iterative solution for it. Subsequently, a tractable and iterative search algorithm is proposed to implement the redundancy resolution of such redundant manipulators. Furthermore, it is shown through simulations that the worst case and average elapsed time that is required to implement the proposed redundancy resolution scheme in a closed-loop implementation is considerably less than that of other numerical optimization methods.**

***Index Terms*—Analytic solution, motion control, parallel robots, redundancy resolution, redundant robots.**

I. Introduction

Although redundancy is a desirable feature in robot manipulators, the presence of redundant actuators will considerably complicate the manipulator control. Since redundancy resolution plays a crucial role in manipulator design and control, the redundancy resolution techniques have been extensively worked out during the past three decades. Despite this long history, previous investigation is often focused on the Jacobian pseudoinverse approach that is proposed originally by Whitny [1] and improved, subsequently, by Liegise [2]. By using the Jacobian pseudoinverse approach, Hollerbach and Suh [3] have suggested methods for minimizations of instantaneous joint torques. Khatib [4] has proposed a scheme to reduce joint torques through inertia-weighted Jacobian pseudoinverse. Dubey and Luh [5] and Chiu [6] used the pseudoinverse approach to optimize the manipulator mechanical advantage and velocity ratio using the force and velocity manipulability ellipsoids. Furthermore, Seraji proposed a configuration-control approach for redundancy resolution of serial manipulators [7]. Some of these methods would automatically generate trajectories that avoid kinematic singularities [8], while other approaches maximize some functions of the joint angles, such as the manipulability measure [9], [10].

Similar to the open-chain serial manipulators, the redundancy resolution of parallel manipulators presents an inherent complexity because of their dynamics constraints [11], particularly when a parallel manipulator is cable driven and the cables' dynamics restriction is added to the manipulator behavior [12]. Some works on redundancy resolution of cable robots are reported based on wrench feasibility of cable robots [13]–[18]. In [13], Bosscher *et al*. present a method for analytical generation of the wrench-feasible workspace boundaries for cable robots. In the numerical algorithm proposed for redundancy resolution by Oh and Agrawal [14], if the tension force of the minimum-norm solution is positive, it is directly applied to the robot. Otherwise, the inputs are all biased to become positive by using the null space contribution of feasible solution. Barrette and Gosselin also showed that it is possible to study redundancy resolution of cable robots as an optimal problem with equality and inequality constraints [15]. In [16], the computation of force distribution is represented as a constrained optimization problem, and different numerical algorithms for force distribution are proposed with respect to their ability to be used in a real-time system. Hassan and Khajepour [17] study actuators' force distribution in a cable robot as a projection problem. In this work, they presented two numerical solutions. A minimum-norm solution is presented first by minimizing the 2-norm of all forces in the cables and redundant limbs, and another solution is given to minimize the 2-norm of just the cable forces. The optimization problem is expressed as a projection on an intersection of convex sets, and Dykstra's projection scheme is used to obtain the solutions. In some recent works, the redundancy resolution of a planar cable robot is studied in the kinematic and dynamic levels [18], in which the redundancy resolution problem is studied as to minimize the 2-norm of actuator forces as well as to minimize the 2-norm of the mobile platform velocity, subject to positive tension in the cables.

It is important to note that, in the implementation of all the aforementioned optimal techniques, numerical methods are the common and the only means to perform the optimization solution [19]. In order to use such techniques in the closed-loop control algorithm, it is required to solve the problem in real time, and therefore, the optimization routine must converge to a solution in a fixed and small period of time. However, this is in direct contrast with generic numerical algorithms, which take a variable step time and exit only when a certain precision has been achieved. The main benefit to have an analytic-iterative solution to the redundancy resolution is to guarantee that the amount of time required for the overall solution remains within an acceptable and small period of time that can be used in real-time implementation of the closed-loop system.

In this paper, an analytic-iterative solution of the redundancy resolution problem and its implementation on a cable-driven redundant manipulator are studied in detail. This is formulated into an optimization problem with equality and inequality constraints. Nonlinear programming techniques, particularly the Karush–Kuhn–Tucker theorem (KKT), are used to analyze this optimization problem and to generate an analytic-iterative solution. Subsequently, a tractable and iterative algorithm is proposed to effectively generate such a solution. It is shown through simulations that the elapsed time that is required to implement the analytic-iterative redundancy resolution scheme is considerably less than that of other numerical optimization methods.

II. Mechanism Origin and Technical Description

To achieve a significant improvement in sensitivity of conventional radio telescope technologies, it is required to drastically increase the instrument's collecting area. A novel concept to increase the collecting area of a single telescope while retaining the functionality of a fully

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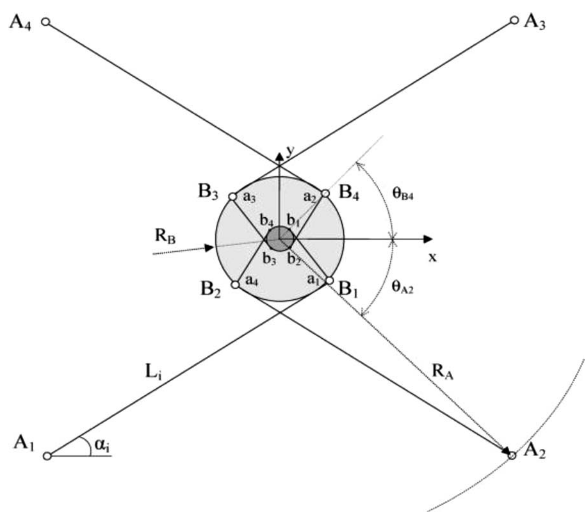

Fig. 1. Schematics of a 2×4RPR mechanism that is employed for the analysis of the LCM/CPM structure.

steerable reflector was devised by researchers at the National Research Council of Canada's Dominion Radio Astrophysical Observatory [20], [21]. The proposed Large Adaptive Reflector (LAR) design is based on two central components: The first is a 200-m reflector which is comprised of actuated panels that form an adjustable paraboloid with a focal length of 500 m and the second is a focal package that is held in place by a multitethered aerostat system. This study focuses exclusively on the analysis of the aerostat positioning system, which uses ground-based winches to actuate the tethers to achieve the telescope's desired workspace.

As is detailed in [22], in order to accurately position the receiver a macro–micro structure is proposed, in which at both macro and micro levels two parallel manipulators with redundant cable-driven actuators are used. In order to keep the analysis complexity at a managing level, while preserving all the important analysis elements, as proposed in [22], a simplified version of the macro–micro structure is considered in the simulation analysis of this paper. This structure is composed of two parallel 4RPR mechanisms, both actuated by four cables. In this simplified structure, although a planar version of the original structure is considered, two important features of the original design namely the actuator redundancy for each subsystem and the macro–micro structure of the original design are employed.

The architecture of the planar macro–micro 2×4RPR parallel manipulator that is considered for our studies is shown in Fig. 1. This manipulator consists of two similar 4RPR parallel structures at macro and micro levels. At each level, the moving platform is supported by four limbs of identical kinematic structure. At macro level, each limb connects the fixed base to the moving platform of the macro manipulator by a revolute joint (R) followed by an actuated prismatic joint (P) and another revolute joint (R). The kinematic structure of a prismatic joint is used to model the elongation of each cable-driven limb. At micro level, similar 4RPR structure is used; however, the base points of the micro manipulators are located on the moving platform of the macro manipulator. Angular positions of base and moving platform attachment points are fully given in [22], and a thorough analysis on the kinematics and dynamics of the macro–micro manipulator is given in [22] and [23]. From the kinematic analysis of this manipulator, the total macro–micro manipulator of the Jacobian matrices $\boldsymbol{J}_t$ is defined as the projection matrices of the total macro- and micro-link velocities $\dot{\mathcal{L}} = [\dot{\boldsymbol{L}}_{\rm macro}, \dot{\boldsymbol{L}}_{\rm micro}]^T$ to the vector of macro and micro moving platform velocities $\dot{\mathcal{X}} = [\dot{\boldsymbol{X}}_{\rm macro}, \dot{\boldsymbol{X}}_{\rm micro}]$ as follows:

$$\dot{\mathcal{L}}_{n\times 1} = (\boldsymbol{J}_t)_{n\times m} \cdot \dot{\mathcal{X}}_{m\times 1}. \tag{1}$$

In fact, the Jacobian matrices $\boldsymbol{J}_t$ are derived by augmenting the macro and micro Jacobian matrices as follows:

$$(\boldsymbol{J}_t)_{n\times m} = \begin{bmatrix} \boldsymbol{J}_M & 0 \\ -\boldsymbol{J}_{Mm} & \boldsymbol{J}_m \end{bmatrix} \tag{2}$$

in which the total Jacobian matrices of the macro–micro manipulator are block triangular matrices, which contain the macro and micro individual Jacobian matrices, i.e., $\boldsymbol{J}_M$ and $\boldsymbol{J}_m$, as the diagonal blocks and the coupling Jacobian matrices $\boldsymbol{J}_{Mm}$ as the off-diagonal block [22].

III. Redundancy Resolution

It can be shown that Jacobian matrices not only reveal the relation between the velocity variables, but also construct the transformation needed to find the actuator forces $\boldsymbol{\tau}_{n\times 1}$ from the forces that act on the moving platform $\boldsymbol{F}_{m\times 1}$ [22]:

$$\boldsymbol{F}_{m\times 1} = (\boldsymbol{J}_t^T)_{m\times n} \cdot \boldsymbol{\tau}_{n\times 1}. \tag{3}$$

Because of the redundancy in actuators, $\boldsymbol{J}_t$ in (2) is a nonsquare matrix with $m < n$. If the manipulator has no redundancy in actuators, the Jacobian matrix $\boldsymbol{J}_t$ would be a square matrix and the actuator forces can be uniquely determined by $\boldsymbol{\tau} = \boldsymbol{J}_t^{-T} \cdot \boldsymbol{F}_{n\times 1}$, provided that $\boldsymbol{J}_t$ is nonsingular. For redundant manipulators, however, there are infinitely many solutions for $\boldsymbol{\tau}$ to be projected into $\boldsymbol{F}_{m\times 1}$. The simplest solution is a minimum-norm solution, which is found from the pseudoinverse of $\boldsymbol{J}_t^T$. The Penrose–Moore pseudoinverse of Jacobian matrices $\boldsymbol{J}$ that is denoted by $\boldsymbol{J}^\dagger$ may be calculated as follows:

$$\boldsymbol{J}^\dagger \triangleq \boldsymbol{J}^T \cdot (\boldsymbol{J} \cdot \boldsymbol{J}^T)^{-1} \quad \text{for} \quad m < n; \quad \boldsymbol{J} \in R^{m\times n}. \tag{4}$$

By this means, the actuator forces can be simply obtained through

$$\boldsymbol{\tau}_{n\times 1} = (\boldsymbol{J}_t^T)^\dagger \cdot \boldsymbol{F}_{m\times 1} \quad \text{for} \quad m < n. \tag{5}$$

This simple solution of the redundancy resolution problem chooses the minimum-norm solution for the actuators among many solutions that satisfy $\boldsymbol{F} = (\boldsymbol{J}_t)^T \cdot \boldsymbol{\tau}$. However, this solution does not ensure positive actuator forces in the cables. Therefore, among the optimal solutions of the redundancy resolution that satisfy the main equality constraint of the projection equation (2), the ones that satisfy the following inequality constraint are of inherent interest for a cable-driven manipulator:

$$\boldsymbol{\tau}_{n\times 1} \geq \boldsymbol{F}_{\rm min}. \tag{6}$$

By choosing $\boldsymbol{F}_{\rm min} \geq \boldsymbol{0}$ as a nonnegative constant, this inequality constraint ensures that the cables are all in tension. In view of this, the redundancy resolution problem for cable-driven manipulators can be reformulated into an *optimization* problem with *equality* and *inequality constraints*. Nonlinear programming methods are used to solve such a problem, and the KKT theorem has served as the basis of those solutions.

A. Karush–Kuhn–Tucker Theorem

Implementation of nonlinear programming theorems, especially the KKT theorem, is directly used in redundancy resolution techniques that are developed for serial redundant manipulators [24]. This optimization problem must satisfy the projection map of (3), as equivalent constraint, and in addition it should provide only the positive tension forces in the cables as the inequality constraint (6). Therefore, it is possible to formulate the redundancy resolution problem as computing the minimum-norm actuator forces

$$\min ||\boldsymbol{\tau}_{n\times 1}||_2$$

under the following two set of constraints:

$$\begin{aligned}\boldsymbol{F}_{m\times 1} &= (\boldsymbol{J}_t^T)_{m\times n}.\boldsymbol{\tau}_{n\times 1},\\ \boldsymbol{\tau}_{n\times 1} &\geq \boldsymbol{F}_{\min}.\end{aligned} \tag{7}$$

If there exists a solution for this optimization problem, it can be written in the following general form:

$$\boldsymbol{\tau}_{n\times 1} = \boldsymbol{F}_0 + \boldsymbol{A}\boldsymbol{y}; \quad \boldsymbol{y} \in R^n \tag{8}$$

in which $\boldsymbol{F}_0$ is the force that is projected by the pseudoinverse of the Jacobian matrix as defined by the following equation and has minimum-norm property, but it is not generally positive:

$$\boldsymbol{F}_0 = (\boldsymbol{J}_t^T)^\dagger \cdot \boldsymbol{F}_{m\times 1}. \tag{9}$$

In this equation, $(\boldsymbol{J}_t^T)^\dagger$ is the pseudoinverse of Jacobian matrix $\boldsymbol{J}_t^T$ defined by (2). Furthermore, denote b as the dimension of the null space of $\boldsymbol{J}_t^T$ and $\boldsymbol{A}$ as the augmented matrix build from orthonormal column vectors of the null space of $\boldsymbol{J}_t^T$:

$$\boldsymbol{A} = \text{orthonormal}\{\text{null space of } (\boldsymbol{J}_t^T)\}. \tag{10}$$

It can be inferred that matrices $\boldsymbol{A}$ may be generated by the collection of the linearly independent column vectors of the real square matrix $\boldsymbol{M} = \boldsymbol{I}_{n\times n} - (\boldsymbol{J}_t^T)^\dagger \cdot \boldsymbol{J}_t^T$, and then make it orthonormal. Returning to the general solution of the redundancy resolution problem, the first term of (8), namely $\boldsymbol{F}_0$, can be viewed as an element of orthogonal complement of the null space of $\boldsymbol{J}_t^T$ (range of $\boldsymbol{J}_t^T$) and the second term $\boldsymbol{Ay}$ as an element of the null space of $\boldsymbol{J}_t^T$. Now, let us rewrite the optimization objective equation (7) as follows:

$$\begin{aligned}||\boldsymbol{\tau}_{n\times 1}||_2^2 &= (\boldsymbol{\tau}_{n\times 1})^T \cdot (\boldsymbol{\tau}_{n\times 1})\\ &= (\boldsymbol{F}_0 + \boldsymbol{A}\boldsymbol{y})^T . (\boldsymbol{F}_0 + \boldsymbol{A}\boldsymbol{y}).\end{aligned} \tag{11}$$

Therefore, (7) can also be represented by

$$\min_y f(\boldsymbol{y}) = (\boldsymbol{F}_0 + \boldsymbol{A}\boldsymbol{y})^T . (\boldsymbol{F}_0 + \boldsymbol{A}\boldsymbol{y}). \tag{12}$$

Under the inequality constraint

$$r(\boldsymbol{y}) = \boldsymbol{F}_{\min} - (\boldsymbol{F_0} + \boldsymbol{A}\boldsymbol{y}) \leq \boldsymbol{0}.$$

Note that, in this formulation, the equality constraint $g(y)$ is always satisfied by choosing proper matrix $\boldsymbol{A}$ as defined in (10). Therefore, from (12) the problem to find minimum-norm and positive tension actuator forces is reduced to minimizing a quadratic function with linear and first-order constraints. To obtain the solution $\boldsymbol{y}$, the KKT theorem can be applied. This is done by defining a function $\varepsilon(\boldsymbol{y}, \boldsymbol{\mu})$ from the corresponding Lagrange multipliers $\boldsymbol{\mu} = [\mu_1, \mu_2, \ldots, \mu_n]^T > 0$ as follows:

$$\varepsilon(\boldsymbol{y}, \boldsymbol{\mu}) = f(\boldsymbol{y}) + \boldsymbol{\mu}^T \cdot r(\boldsymbol{y}). \tag{13}$$

From the KKT theorem, a necessary condition can be derived for $\boldsymbol{y}_0$ to yield to a local minimum of $f(\boldsymbol{y})$ under the condition $r(\boldsymbol{y}) \leq 0$. This condition is $\boldsymbol{\mu}^0 \geq 0$ ($\boldsymbol{\mu}^0 = [\mu_1^0, \mu_2^0, \ldots, \mu_n^0]$, $\mu_i^0 \geq 0$) that satisfies the following equations:

$$\begin{aligned}&\frac{\partial}{\partial \boldsymbol{y}}\varepsilon(\boldsymbol{y}, \boldsymbol{\mu})\,|_{\substack{\boldsymbol{y}=\boldsymbol{y_0}\\ \boldsymbol{\mu}=\boldsymbol{\mu_0}}} =\\ &\frac{\partial}{\partial \boldsymbol{y}}\left(f(\boldsymbol{y}) + \boldsymbol{\mu}^T \cdot r(\boldsymbol{y})\right)|_{\substack{\boldsymbol{y}=\boldsymbol{y_0}\\ \boldsymbol{\mu}=\boldsymbol{\mu_0}}} = \boldsymbol{0}\end{aligned} \tag{14}$$

in which $\boldsymbol{y}_0$ is a stationary point. Furthermore, the following condition must be also satisfied:

$$\boldsymbol{\mu}^T \cdot r(\boldsymbol{y})|_{\boldsymbol{y}=\boldsymbol{y_0}} = \boldsymbol{0}. \tag{15}$$

Substitution of $f(\boldsymbol{y})$ and $r(\boldsymbol{y})$ into (13) gives

$$\begin{aligned}\varepsilon(\boldsymbol{y}, \boldsymbol{\mu}) &= \boldsymbol{F}_0^T\boldsymbol{F}_0 + \boldsymbol{F}_0^T\boldsymbol{A}\boldsymbol{y} + \boldsymbol{y}^T\boldsymbol{A}^T\boldsymbol{F}_0 + \boldsymbol{y}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{y}\\ &\quad + \boldsymbol{\mu}^T(\boldsymbol{F}_{\min} - \boldsymbol{F}_0 - \boldsymbol{A}\boldsymbol{y}).\end{aligned} \tag{16}$$

Differentiate the aforementioned equation and simply to

$$\begin{aligned}&\frac{\partial}{\partial \boldsymbol{y}}\varepsilon(\boldsymbol{y}, \boldsymbol{\mu})|_{\boldsymbol{y}=\boldsymbol{y_0}} =\\ &2\boldsymbol{F}_0^T\boldsymbol{A} + 2\boldsymbol{y}_0^T\boldsymbol{A}^T\boldsymbol{A} - \boldsymbol{\mu}^T\boldsymbol{A} = \boldsymbol{0}.\end{aligned} \tag{17}$$

Substitution of $r(\boldsymbol{y})$ into (15) gives

$$\boldsymbol{\mu}^T(\boldsymbol{F}_{\min} - \boldsymbol{F}_0 - \boldsymbol{A}\boldsymbol{y_0}) = 0. \tag{18}$$

Writing (17) and (18) together, we reach to

$$\begin{cases}2\boldsymbol{F}_0^T\boldsymbol{A} + 2\boldsymbol{y}_0^T\boldsymbol{A}^T\boldsymbol{A} - \boldsymbol{\mu}^T\boldsymbol{A} = \boldsymbol{0}\\ \boldsymbol{\mu}^T(\boldsymbol{F}_{\min} - \boldsymbol{F}_0 - \boldsymbol{A}\boldsymbol{y}_0) = 0.\end{cases} \tag{19}$$

In this set of nonlinear equations, $\boldsymbol{F}_0$ and $\boldsymbol{A}$ are known from (9) and (10), respectively. Note that the last equation of (19) is a nonlinear equation, and it may lead to multiple solutions for this set of equations. By solving this set of equations, $\boldsymbol{y}_0$ and $\boldsymbol{\mu}^T$ vectors are obtained. However, only the set of solution that satisfy the KKT theorem condition $\boldsymbol{\mu} = [\mu_1, \mu_2, \ldots, \mu_n]^T \geq 0$ are acceptable. If there is no set of solution with positive $\boldsymbol{\mu}$, the optimization problem does not lead to any solution. Moreover, it is well known that the KKT theorem provides only the necessary condition to derive the local minimum of the optimization problem. In order to have sufficient condition for the solution and to ensure that the solution is not local but global, the convexity of the Lagrangian function $\varepsilon(\boldsymbol{y}, \boldsymbol{\mu})$ must be analyzed. This analysis is done in Section III-B.

B. Lagrangian Function Convexity

By substituting $r(\boldsymbol{y})$ into the Lagrangian function detailed in (16) and using the fact that $\boldsymbol{F}_0$ and $\boldsymbol{Ay}$ are orthogonal to each other, this function is simplified to

$$\begin{aligned}\varepsilon(\boldsymbol{y}, \boldsymbol{\mu}) &= \boldsymbol{y}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{y} + (\boldsymbol{F}_0^T\boldsymbol{A} - \mu^T\boldsymbol{A})\boldsymbol{y}\\ &\quad + \boldsymbol{y}^T\boldsymbol{A}^T\boldsymbol{F}_0 + \boldsymbol{F}_0^T\boldsymbol{F}_0 + \boldsymbol{\mu}^T(\boldsymbol{F}_{\min} - \boldsymbol{F}_0).\end{aligned} \tag{20}$$

Equation (20) represents a quadratic form for the Lagrangian function $\varepsilon(\boldsymbol{y}, \boldsymbol{\mu})$. Convexities of quadratic functions are guaranteed provided that the second-order term of the quadratic form is positive definite. In such case, the solution of optimal problem is globally optimal. However, if this term is positive semidefinite, the quadratic function has an infinite number of local minimums. The second-order term of (20) is as follows:

$$\boldsymbol{A}^T\boldsymbol{A}. \tag{21}$$

Thus, sufficient condition for regular point $\boldsymbol{y}_0$ to be a global minimum of the function is that $\boldsymbol{A}^T\boldsymbol{A}$ is positive definite. As we explained earlier in (10), the matrix $\boldsymbol{A}$ is generated by collecting and orthonormalizing the linearly independent column vectors of $\boldsymbol{I}_{n\times n} - (\boldsymbol{J}_t^T)^\dagger \cdot \boldsymbol{J}_t^T$. Therefore, column vectors of $\boldsymbol{A}$ are linearly independent, and each column vector of $\boldsymbol{A}$ is orthogonal to other columns, and therefore

$$\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}_{n\times n}. \tag{22}$$

Therefore, $\boldsymbol{A}^T\boldsymbol{A}$ is always positive definite, and the quadratic Lagrangian function given in (20) is always convex. Through this analysis, the sufficient condition for regular point $\boldsymbol{y}_0$ to be a global minimum is established.

IV. Analytic-Iterative Solution to the Optimization Problem

In this section, a procedure is given to generate the solution of the optimization problem in an analytical way. Note that $\varepsilon(\boldsymbol{y},\boldsymbol{\mu})$ is convex; $\boldsymbol{A}^T\boldsymbol{A}=\boldsymbol{I}$ becomes positive definite and, therefore, nonsingular. Rewrite (19) by substitution of $\boldsymbol{A}^T\boldsymbol{A}=\boldsymbol{I}$ as

$$\begin{cases} 2\boldsymbol{F}_0^T\boldsymbol{A}+2\boldsymbol{y}_0^T-\boldsymbol{\mu}^T\boldsymbol{A}=\boldsymbol{0} \\ \boldsymbol{\mu}^T(\boldsymbol{F}_{\min}-\boldsymbol{F}_0-\boldsymbol{A}\boldsymbol{y}_0)=0. \end{cases} \tag{23}$$

Use the transpose of the first equation as follows:

$$\begin{cases} 2\boldsymbol{A}^T\boldsymbol{F}_0+2\boldsymbol{y}_0-\boldsymbol{A}^T\boldsymbol{\mu}=\boldsymbol{0} \\ \boldsymbol{\mu}^T(\boldsymbol{F}_{\min}-\boldsymbol{F}_0-\boldsymbol{A}\boldsymbol{y}_0)=0. \end{cases} \tag{24}$$

Equation (18) is calculated as follows:

$$\sum_{i=1}^{n}\mu_i r_i(y_0)=0. \tag{25}$$

Since $\mu_i \geq 0$ and from (15) $r_i(\boldsymbol{y}_0)\leq 0$, then (25) yields

$$\begin{cases} \mu_i=0 \text{ for } r_i(\boldsymbol{y}_0)<0 \\ \mu_i\geq 0 \text{ for } r_i(\boldsymbol{y}_0)=0. \end{cases} \tag{26}$$

This equation implies that the inequality $\mu_i>0$ holds only at instances, where the following equality constraints hold: $r_i(\boldsymbol{y})=(\boldsymbol{F}_{\min})_i-(\boldsymbol{\tau}_{n\times 1})_i=0$ or $(\boldsymbol{\tau}_{n\times 1})_i=(\boldsymbol{F}_{\min})_i$. In fact, in this case, the specific actuator forces lie at the boundary of the inequality constraint and should be set by the constant $(\boldsymbol{F}_{\min})_i$. Furthermore, the condition $\mu_i=0$ is satisfied only when the inequality constraint $r_i(\boldsymbol{y})=(\boldsymbol{F}_{\min})_i-(\boldsymbol{\tau}_{n\times 1})_i<0$ holds. In other words, the actuator force lies inside the solution set defined by this inequality constraint. Considering these facts, the solution of the optimization problem can be derived from three different cases.

A. Case 1

Assume that all forces are inside the solution set which is defined by the inequality constraint and $\mu_i=0$. Hence, in this case, $r_i(\boldsymbol{y})>0$ for all $i=1,2,\ldots,n$, and (24) is simplified to

$$-2\boldsymbol{y}_0=2\boldsymbol{A}^T\boldsymbol{F}_0\rightarrow\boldsymbol{y}_0=-\boldsymbol{A}^T\boldsymbol{F}_0. \tag{27}$$

Equation (27) can be written as a set of linear equations in the following matrices form:

$$[\boldsymbol{B}_0]_{n\times n}\cdot[\boldsymbol{X}_0]_{n\times 1}=[\boldsymbol{C}_0]_{n\times 1} \tag{28}$$

in which

$$\boldsymbol{B}_0=-2\boldsymbol{I}_{n\times n},\quad \boldsymbol{X}_0=\boldsymbol{y}_{0_{n\times 1}},\quad \boldsymbol{C}_0=\left[2\boldsymbol{A}^T\boldsymbol{F}_0\right]_{n\times 1}. \tag{29}$$

These linear equations can be easily solved, and the value of $\boldsymbol{y}_0$ be obtained as given in (27), with the assumption applies in this case that $\boldsymbol{\mu}=[0,\ldots,0]^T$. Therefore,

$$\boldsymbol{\tau}_{n\times 1}=\boldsymbol{F}_0+\boldsymbol{A}\boldsymbol{y}_0. \tag{30}$$

It is clear that the matrix $\boldsymbol{B}_0$ in (29) is always invertible.

B. Case 2

Consider the case in which for the optimal solution $\mu_i>0$ for some is within $[1,2,\ldots,n]$, and $\mu_j=0$ for the rest of them, namely, $j\neq i$ and $j=[1,2,\ldots,n]$. In this case, for the elements in which $\mu_i>0$, the corresponding forces τ_i's are obtained from

$$\boldsymbol{\tau}_i=[\boldsymbol{F}_0+\boldsymbol{A}\boldsymbol{y}_0]_i=[\boldsymbol{F}_{\min}]_i \tag{31}$$

and for the rest of μ_js their corresponding forces are calculated from an equation deduced from (24) by the elimination of the rows and columns of matrices $\boldsymbol{A}$ related to the zero μ_i's. Therefore, $\boldsymbol{y}_0$, and $\boldsymbol{\mu}$ can be obtained by solving the following linear equations:

$$\begin{bmatrix} \boldsymbol{B}_0 & [\vec{a}_1^T \ \ldots \ \vec{a}_j^T] \\ \begin{bmatrix}\vec{a}_1\\ \vdots \\ \vec{a}_j\end{bmatrix} & \boldsymbol{0} \end{bmatrix}\cdot\begin{bmatrix}\boldsymbol{y}_0 \\ \begin{bmatrix}\mu_1\\ \vdots\\ \mu_j\end{bmatrix}\end{bmatrix}=\begin{bmatrix}2\boldsymbol{A}^T\boldsymbol{F}_0 \\ \begin{bmatrix}\boldsymbol{F}_{\min_1}-\boldsymbol{F}_{0_1}\\ \vdots \\ \boldsymbol{F}_{\min_j}-\boldsymbol{F}_{0_j}\end{bmatrix}\end{bmatrix}. \tag{32}$$

In this equation, we suppose that

$$\boldsymbol{A}=\begin{bmatrix}\vec{a}_1\\ \vdots\\ \vec{a}_n\end{bmatrix}_{n\times 1}$$

and each $\vec{a}_j$ is a row vector, and only the row vectors corresponding to the nonzero μ_i's are left in this equation. It is important to note that always the left-hand side matrix $\boldsymbol{B}$ in (32) is always invertible. The proof is given in the Appendix.

C. Case 3

Assume that all forces are on the boundary of the inequality constraints, and $\mu_i\geq 0$. Therefore, $r_i(\boldsymbol{y})=0$ for all $i=1,2,\ldots,n$, and thus, the optimal actuator forces for all joints are obtained from

$$\boldsymbol{\tau}_{n\times 1}=\boldsymbol{F}_0+\boldsymbol{A}\boldsymbol{y}_0=\boldsymbol{F}_{\min}. \tag{33}$$

In other words, in this case, the solution is known, and therefore, it is not necessary to find $\boldsymbol{y}_0$ from (24). Finally, assuming that $\boldsymbol{F}_{\min}\geq 0$ and the fact that $\varepsilon(\boldsymbol{y},\boldsymbol{\mu})$ is convex, there is a unique minimum solution for this problem that can be found from the three cases. The unique solution can, therefore, be found through a search procedure that is detailed in Section IV-A.

D. Search Algorithm

This section is devoted to develop a search routine for the proposed algorithm given for the redundancy resolution solution. In this search routine and in the first loop, we assume that all forces lie on the boundary of solution set which is defined by the inequality constraints, i.e., $\forall\mu_i>0$, and therefore, we know that the solution and all forces equal to $F_{\min}$ by (33). If this solution can satisfy all the optimization conditions, i.e., $r_i(y)\leq 0\ \forall i$, then it is a valid solution for the optimization problem and the search algorithm is terminated. Otherwise, the combinations of forces that can lie on the boundaries of the inequality constraints must be checked and found. For this search, there are $\binom{m}{S}$ combinations with $m=[1,2,\ldots,n]^T$ and $S=1$ to n, which might be a plausible solution for the problem. These solutions can be checked by sweeping all possible combinations through changing S in a loop. However, only a solution to the optimization problem is acceptable if it satisfies all the optimization constraints, i.e., $r_i(\boldsymbol{y})\leq 0,\ \forall i$. As

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shown in Fig. 2, the last loop is in fact the implementation of case 1, in which all forces are inside the boundary, or in other words, all forces are larger than $F_{\min}$. Note that, in this algorithm, we solve the set of equations corresponding to the KKT theorem condition ($\forall \mu_i \geq 0$), and there is no need to recheck this condition. Therefore, the search algorithm can be summarized as follows. Denote $\boldsymbol{\mu} = [\mu_1, \mu_2, \ldots, \mu_n]$; then the following steps will execute.

Step 0. Set $S = 0$.

Step 1. Assume S forces are inside the boundary of the inequality constraints ($\mu_i > 0$) and $n - S$ forces are on the boundary ($\mu_i = 0$). Find the possible combinations of the forces that lie on the boundaries of the inequality constraints by

$$\text{combinations} \begin{pmatrix} \boldsymbol{M} \\ S \end{pmatrix} \text{ and } \boldsymbol{M} = [1, 2, \ldots, n]^T.$$

Step 2. For each combination

1) For each i that $\mu_i = 0$, compute $\boldsymbol{y}_0$ and corresponding forces from

$$\tau_i = [\boldsymbol{F}_{\min}]_i .$$

2) For each j that $\mu_j > 0$, eliminate the rows and columns of matrices $\boldsymbol{B}$ and corresponding elements of vectors $\boldsymbol{X}$ and, $\boldsymbol{C}$ related to the zero of μ_i's and compute $\boldsymbol{y}_0$, and the rest of μ_j's by solving the linear equations given in (32). Then, compute corresponding forces as follows:

$$\boldsymbol{\tau}_j = [\boldsymbol{F}_0 + \boldsymbol{A}\boldsymbol{y}_0]_j .$$

Augment μ_i's and μ_j's to generate μ, and consider the set of solution $[\boldsymbol{y}_0, \boldsymbol{\mu}]$.

Step 3. Check if all derived μ_j's that satisfy$\mu_i \geq 0$ are also satisfying $r_i(\boldsymbol{y}) \leq 0$ condition. If this is true, this set of solution is the optimal solution and stop the search algorithm; otherwise, continue.

Step 4. If $S < m$, then set $S = S + 1$, and *go to* Step 1.

Step 5. If $S = m$, then the optimization problem does not have any solution and the resultant forces $\boldsymbol{\tau}_{n\times 1}$ cannot be generated under these constraints. Finally, we can represent this solution as an optimal projection map that projects forces from Cartesian space into joint space.

It can be shown that if there exists a solution, the number of iterations to perform in order to find the solution lies between

$$1 \leq \text{Number of iterations} \leq \sum_{s=0}^{n} \begin{pmatrix} n \\ s \end{pmatrix}. \tag{34}$$

For example, for our case study, the cable driven redundant manipulator with six degrees of freedom and two degree of redundancy, we have $1 \leq \text{Number of iterations} \leq 256$.

V. Implementation Results

In this section, the developed redundancy resolution technique is implemented on the cable-driven redundant manipulator that is described in Section II, and its performance in the closed-loop control system is evaluated. The block diagram of closed-loop simulations is given in Fig. 3, in which a simple inverse dynamics control in addition to a decentralized Proportional-Derivative (PD) controller is used for the closed-loop system. By proper tuning of the controller gains, this control topology is capable of providing the required tracking performance, despite the actuator saturation limits. For the sake of comparison, different redundancy resolution schemes, which include the proposed analytic-iterative method and three other numerical methods,

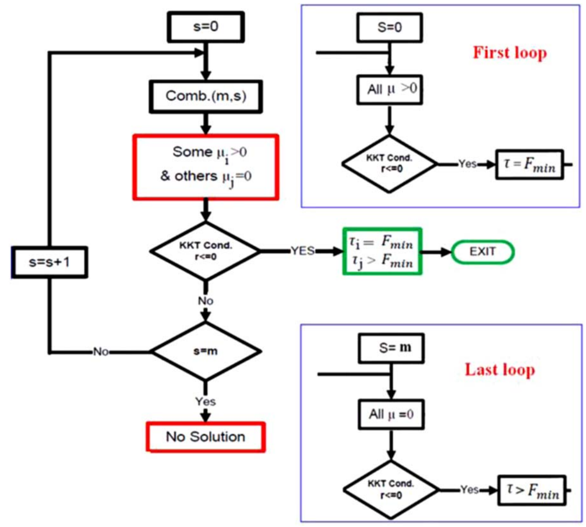

Fig. 2. Flowchart of the search algorithm.

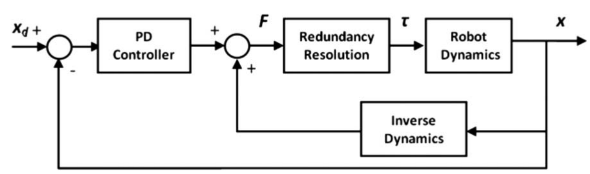

Fig. 3. Block diagram of the closed-loop control topology using an inverse dynamics control in addition to a decentralized PD controller.

are used as the redundancy resolution engines. The numerical methods used here are trust-region-reflective [25], [26], active-set [27], and interior-point optimization [28]. All the methods are simulated using MATLAB and are computed in a PC with a "core2 Duo E4300 (1.8 GHz)" CPU and 2-GB RAM. In order to implement this comparison, MATLAB fmincon function has been used that allows applying these three numerical-iterative methods in the MATLAB environment. The proposed analytic-iterative method is also implemented in MATLAB.

A typicaltrajectory of the macro manipulator in the Cartesian space is considered for the simulations, in which the x and y motion of the platform is considered to be 20 m, while the orientation changes about 45° within 600 s. It can be shown that the motion of this manipulator in closed loop well tracks the desired trajectory. Since the main purpose of this paper is to evaluate the redundancy resolution technique, and not the control structure itself, the composition of the cable forces to achieve such performance is shown in Fig. 4. As is clearly seen in Fig. 4, all the cables remain in tension, and the forces are always higher than the minimum tension force threshold, i.e., set to $F_{\min} = 10$ N in these simulations. It is clearly seen from Fig. 4 that the proposed redundancy resolution routine is capable of projecting the required Cartesian forces into all tension forces in the cables. Fig. 5 illustrates the elapsed time that is required to calculate the redundancy resolution scheme through different methods. As is seen in Fig. 5, the worst case elapsed time that is required to perform our proposed method is about 3.8 ms, which is much less than that of the other numerical optimization methods. Furthermore, the average elapsed time that is required to perform the proposed method at each step is also much better than that of the other methods. Moreover, in the numerical algorithms that are simulated for this case, the elapsed time is significantly varying in different iterations, and, in some instances, there are abrupt changes in their variation. This is because of the fact that in these algorithms the number of iterations

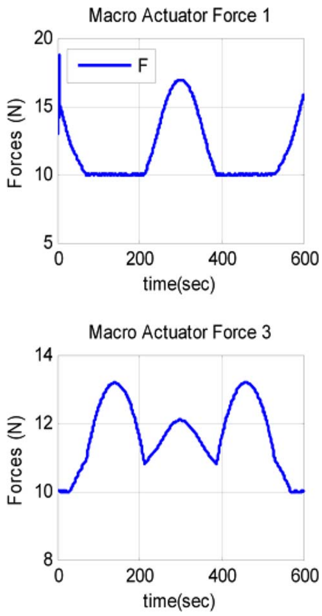

Fig. 4. Tensions in cables at macro level.

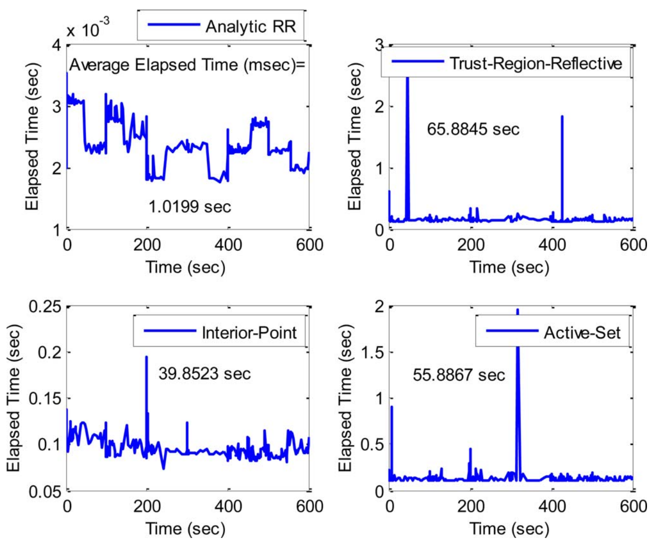

Fig. 5. Total elapsed time to calculate optimal forces in analytic-iterative redundancy resolution and other numerical optimizations methods at all iterations.

greatly depends on the robot configurations. In order to quantitatively compare different routines, the average elapsed time that is used in different redundancy resolution routines is given in Table I. To track a desired 600-s trajectory, it is seen that the average elapsed time in our proposed analytical method is about 1.02 s, which is more than 39 times shorter than the interior-point method, and is the fastest scheme among all analyzed routines. This benefit is much more appreciated comparing the result with the trust-region-reflective optimization method, which is more than 65 times slower than our proposed method.

In order to compare our proposed method by another means with the best performed numerical method, namely the interior point method, the number of iterations that is required at each step to reach to the solution is illustrated in Fig. 6. This figure indicates that for the desired trajectory, in the worst case the solution is reached after 30 iterations

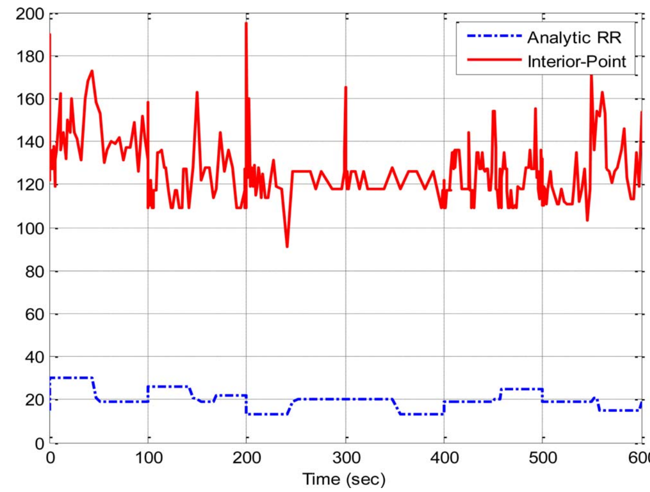

Fig. 6. Number of iterations at each step performed in the proposed method and that in the interior point algorithm.

TABLE I
AVERAGE ELAPSED TIME TO EXECUTE REDUNDANCY RESOLUTION

Algorithms	Average elapsed time at each iteration (msec)	Speed Comparison
Proposed Method	1.0199	1
Trust-Region-Reflective	65.8845	65
Interior-Point	39.8523	39
Active-Set	55.8867	55

($S = 2$), while this is about 200 iterations for the interior point method, which is the best among the numerical-iterative methods. This result confirms the superior performance of the proposed method and its suitability for its future online implementations.

VI. CONCLUSION

In this paper, an analytic-iterative solution for the redundancy resolution problem is proposed, and its implementation on a cable-driven redundant manipulator is studied in detail. This task is formulated into an optimization problem with inequality constraints. Nonlinear programming techniques, particularly the KKT theorem, are used to analyze this optimization problem and to generate the analytic solution. Subsequently, a tractable search algorithm is proposed to effectively search for the conditions that such a solution may exist. It is shown through simulations that the average elapsed time of the proposed redundancy resolution scheme in the closed-loop structure is considerably less than those of other numerical-iterative optimization methods.

APPENDIX

The following proof shows that matrix $\boldsymbol{B}$ in (32) is always invertible. For this purpose, let us calculate the determinant of this matrix. Note that if $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C},$ and $\boldsymbol{D}$ are matrices with proper dimensions augmented into the following form and if $\boldsymbol{A}$ is invertible, the determinant of the augmented matrix can be written as follows:

$$\det\left(\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}\right) = \det(\boldsymbol{A}) \det\left(\boldsymbol{D} - \boldsymbol{C}A^{-1}\boldsymbol{B}\right). \quad (35)$$

The determinant of the $\boldsymbol{B}$ matrix in (32) reads

$$\det\left(\begin{bmatrix} -2\boldsymbol{I}_{n\times n} & \begin{bmatrix}\vec{a}_1^T & \dots & \vec{a}_j^T\end{bmatrix} \\ \begin{bmatrix}\vec{a}_1 \\ \vdots \\ \vec{a}_j\end{bmatrix} & 0 \end{bmatrix}\right).$$

We suppose that

$$\boldsymbol{A}' = \begin{bmatrix}\vec{a}_1 \\ \vdots \\ \vec{a}_j\end{bmatrix}$$

and each $\vec{a}_j$ is a row vector, and only the row vectors that correspond to the nonzero μ_i's are left in this equation. Use (35) to calculate this determinant

$$|\boldsymbol{B}| = (-2)^n |\boldsymbol{I}_{n\times n}||\boldsymbol{0}_{j\times j} - \boldsymbol{A}'(-2\boldsymbol{I}_{n\times n})^{-1}\boldsymbol{A}'^T| = (-2)^{n+j}|\boldsymbol{A}'A'^T|.$$

The vectors of matrices $\boldsymbol{A}$ are orthonormal; therefore

$$\{\vec{\boldsymbol{a}}_i\vec{\boldsymbol{a}}_j^T = 1 \text{ when } i=j \text{ and } \vec{\boldsymbol{a}}_i\vec{\boldsymbol{a}}_j^T = \boldsymbol{0} \text{ when } i \neq j\} \Rightarrow \boldsymbol{A}'A'^T = \boldsymbol{I}_{j\times j}$$

$$|\boldsymbol{B}| = (-2)^{n+j}.$$

Therefore, the matrix $\boldsymbol{B}$ is always invertible.

References

[1] D. E. Whitney, "Resolved motion rate control of manipulators and human prostheses," *IEEE Trans. Man-Mach. Syst.*, vol. MMS-10, no. 2, pp. 47–53, Jun. 1969.

[2] A. Liegeois, "Automatic supervisory control of the configuration and behavior of multibody mechanisms," *IEEE Trans. Syst., Man Cybern.*, vol. SMC-7, no. 12, pp. 868–871, Dec. 1977.

[3] J. M. Hollerbach and K. C. Suh, "Redundancy resolution of manipulators through torque optimization," *IEEE J. Robot. Autom.*, vol. RA-3, no. 4, pp. 308–316, Aug. 1987.

[4] O. Khatib, "Unified approach for motion and force control of robot manipulators: The operational space formulation," *IEEE J. Robot. Autom.*, vol. RA-3, no. 1, pp. 43–53, Feb. 1987.

[5] R. Dubey and J. Y. S. Luh, "Redundant robot control using task based performance measures," *J. Robot. Syst.*, vol. 5, no. 5, pp. 409–432, 1988.

[6] S. L. Chiu, "Task compatibility of manipulator postures," *Int. J. Robot. Res.*, vol. 7, no. 5, pp. 13–21, 1988.

[7] H. Seraji, "Configuration control of redundant manipulators: Theory and implementation," *IEEE Trans. Robot. Autom.*, vol. 5, no. 4, pp. 472–490, Aug. 1989.

[8] D. P. Marin, J. Baillieul, and J. Hollerbach, "Resolution of kinematic redundancy using optimization techniques," *IEEE Trans. Robot. Autom.*, vol. 5, no. 4, pp. 529–533, Aug. 1989.

[9] T. Yoshikawa, "Manipulability of robotic mechanisms," *Int. J. Robot. Res.*, vol. 4, no. 2, pp. 3–9, Summer 1985.

[10] S. Soylu, B. J. Buckham, and R. P. Podhorodeski, "Dexterous task-priority based redundancy resolution for underwater-manipulator systems," *Trans. Can. Soc. Mech. Eng.*, vol. 31, no. 4, pp. 519–533, 2007.

[11] S.-H. Cha, T. A. Lasky, and S. A. Velinsky, "Kinematic redundancy resolution for serial-parallel manipulators via local optimization including joint constraints," *Mechanics Based Design Struct. Mach.*, vol. 34, no. 2, pp. 213–239, Jul. 2006.

[12] Y. Babazadeh Bedoustani, H. D. Taghirad, and M. M. Aref, "Dynamics analysis of a redundant parallel manipulator driven by elastic cables," in *Proc.10th Int. Conf. Control, Automation and Vision*, Hanoi, Vietnam, Oct. 2008, pp. 536–542.

[13] P. Bosscher, A. Riechel, and I. Ebert-Uphoff, "Wrench-feasible workspace generation for cable-driven robots," *IEEE Trans. Robot.*, vol. 22, no. 5, pp. 890–902, Oct. 2006.

[14] S.-R. Oh and S. K. Agrawal, "Cable suspended planar robots with redundant cables: Controllers with positive tensions," *IEEE Tran. Robot.*, vol. 21, no. 3, pp. 457–465, Jun. 2005.

[15] G. Barrette and C. Gosselin, "Determination of the dynamic workspace of cable-driven planar parallel mechanisms," *J. Mech. Design*, vol. 127, no. 2, pp. 242–248, 2005.

[16] T. Bruckmann, A. Pott, and M. Hiller, "Calculating force distributions for redundantly actuated tendon-based Stewart platforms," *Advances in Robot Kinematics: Mechanisms and Motion*, Amsterdam, The Netherlands: Springer, 2006, pp. 403–412.

[17] M. Hassan and A. Khajepour, "Optimization of actuator forces in cable-based parallel manipulators using convex analysis," *IEEE Trans. Robot.*, vol. 24, no. 3, pp. 736–740, Jun. 2008.

[18] M. Agahi and L. Notash, "Redundancy resolution of wire-actuated parallel manipulators," *Trans. Can. Soc. Mech. Eng.*, vol. 33, no. 4, 2009.

[19] B. Zi, B. Y. Duan, J. L. Du, and H. Bao, "Dynamic modeling and active control of a cable-suspended parallel robot," *Mechatronics*, vol. 18, no. 1, pp. 1–12, Feb. 2008.

[20] B. Carlson, L. Bauwens, L. Belototski, E. Cannon, Y. Deng, P. Dewdney, J. Fitzsimmons, D. Halliday, K. Krschner, G. Lachapelle, D. Lo, P. Mousavi, M. Nahon, L. Shafai, S. Stiemer, R. Taylor, and B. Veidt, "The large adaptive reflector: A 200-m diameter, wideband, cm-wave radio telescope," *Proc. SPIE*, vol. 4015, pp. 33–44, 2000.

[21] T. Legg, "A proposed new design for a large radio telescope," *Astron. Astrophys. Suppl. Ser.*, vol. 130, no. 4, pp. 369–379, 1998.

[22] H. Taghirad and M. Nahon, "Kinematic analysis of a macro-micro redundantly actuated parallel manipulator," *Adv. Robot.*, vol. 22, no. 2, pp. 657–687, Jul. 2008.

[23] H. Taghirad and M. Nahon, "Dynamic analysis of a macro-micro redundantly actuated parallel manipulator," *Adv. Robot.*, vol. 22, no. 3, pp. 949–981, Sep. 2008.

[24] Y. Nakamura, *Advanced Robotics: Redundancy and Optimization*. Reading, MA: Addison-Wesley, 1991.

[25] T. F. Coleman and Y. Li, "An interior, trust region approach for nonlinear minimization subject to bounds," *SIAM J. Optim.*, vol. 6, pp. 418–445, 1996.

[26] M. J. D. Powell, "Trust region algorithm for equality constrained optimization," *Math. Program., Ser. B*, vol. 49, no. 2, pp. 189–211, Dec. 1990.

[27] W. W. Hager and H. Zhang, "A new active set algorithm for box constrained optimization," *J. Optim.*, vol. 17, no. 2, pp. 526–557, 2006.

[28] S. Mehrotra and M. G. Ozevin, "On the implementation of interior point decomposition algorithms for two-stage stochastic conic programs," *SIAM J. Optim.*, vol. 19, no. 4, pp. 1846–1880, 2009.