Solution1:

1. Modeling the system:

Mechanical model:

$$m\ddot{x} = W - F_E = mg - K\frac{i^2}{(x+d)^2}$$

Electrical model:

$$E = R_i i + L\frac{di}{dt} + R_s i \rightarrow \frac{di}{dt} = \frac{1}{L}(E - (R_i + R_s)i)$$

Also:

$$x = [x \quad \dot{x} \quad i]^T = [x_1 \quad x_2 \quad x_3]^T$$

$$u = E$$

So we have:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_1 = g - \frac{K}{m}\frac{x_3^2}{(x_1+d)^2}$$

$$\dot{x}_3 = \frac{1}{L}(u - (R_i + R_s)x_3)$$

2. Equilibrium point $(\boldsymbol{x^*}, \boldsymbol{u^*})$

$$\dot{x}_1 = x_2 = 0 \rightarrow \boxed{x_2^* = 0}$$

$$\dot{x}_3 = \frac{1}{L}(u - (R_i + R_s)x_3) = 0 \rightarrow \boxed{x_3^* = \frac{u^*}{R_i + R_s}}$$

$$\dot{x}_2 = g - \frac{K}{m}\frac{x_3^2}{(x_1+d)^2} = 0 \rightarrow x_3^* = \sqrt{\frac{mg}{K}}(x_1^* + d)$$

$$\rightarrow \boxed{x_1^* = \sqrt{\frac{K}{mg}}x_3^* - d}$$

3. **Stability type:**

$$A=\frac{\partial f(x)}{\partial x}=\begin{bmatrix}0 & 1 & 0\\ \frac{2Kx_3^{*2}}{m(x_1^*+d)^3} & 0 & -\frac{2Kx_3^*}{m(x_1^*+d)^2}\\ 0 & 0 & -\frac{(R_i+R_s)}{L}\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ \frac{2g}{x_1^*+d} & 0 & -\sqrt{\frac{gK}{m}}\frac{2}{(x_1^*+d)}\\ 0 & 0 & -\frac{(R_i+R_s)}{L}\end{bmatrix}$$

Where $\frac{\partial f(x)}{\partial x}$ can be simplified as:

$$A=\frac{\partial f(x)}{\partial x}=\begin{bmatrix}0 & 1 & 0\\ a_{21} & 0 & -a_{23}\\ 0 & 0 & -a_{33}\end{bmatrix}$$

So:

$\det(\lambda I-A)=0\rightarrow(\lambda+a_{33})(\lambda^2-a_{21})=0$

With these values:

$R_i=10\ ohm\ , R_s=1\ ohm\ , L=0.4125, g=9.81\ m/\ s^2\ , x_1^*=7.5mm$

Therefore:

$$\lambda_{12}=\pm\sqrt{a_{21}}=\pm\sqrt{\frac{2g}{x_1^*+d}}=\pm 36.3\ \ Sadlle\ Point$$

$$\lambda_{\#}=-a_{33}=-26.7\ Stable\ Node$$

Solution2:

$\mathrm{f}_1(x_1^*,x_2^*)=0\quad x_1^2x_2+x_1x_2^2=0\quad(1)$

$\mathrm{f}_2(x_1^*,x_2^*)=0\quad(-x_1+x_2)(x_2^2-1)=0\qquad(2)\quad\Longrightarrow x_2=x_1\ ,\quad x_2=1\ ,\quad x_2=-1$

$if\ x_2=x_1\ \Longrightarrow(1)\ \Longrightarrow\ 2x_1^3=0\ \Longrightarrow\ (x_1^*,x_2^*)=(0,0)$

$if\ x_2=1\ \Longrightarrow(1)\ \Longrightarrow\ x_1^2+x_1=0\Longrightarrow\ (x_1^*,x_2^*)=(0,1)\ \&\ (-1,1)$

$if\ x_2 = -1 \implies (1) \implies -x_1^2 + x_1 = 0 \implies (x_1^*, x_2^*) = (0,-1)\ \&\ (1,-1)$

$$A = \frac{\partial f}{\partial x}|x = x^* = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1x_2 + x_2^2 & x_1^2 + 2x_1x_2 \\ -x_2^2 + 1 & -2x_1x_2 + 3x_2^2 - 1 \end{bmatrix}$$

$A = \frac{\partial f}{\partial x}|x^* = (0,0) = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \implies \lambda_1 = -1\ \&\ \lambda_2 = 0 \implies$ *Multiple but non-isolated*

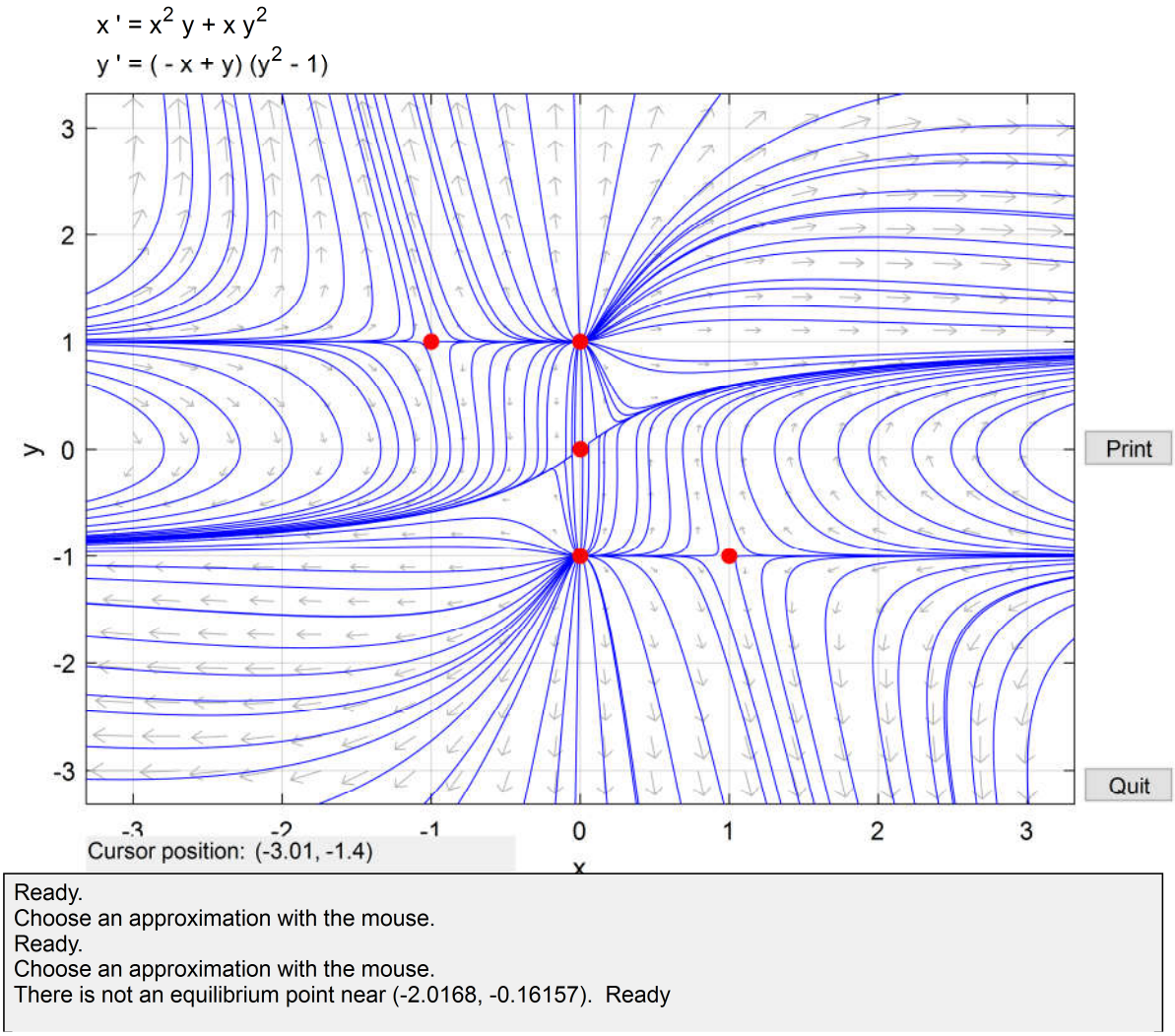
$A = \frac{\partial f}{\partial x}|x^* = (0,1) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \implies \lambda_1 = 1\ \&\ \lambda_2 = 2 \implies$ *Unstable Node*

$A = \frac{\partial f}{\partial x}|x^* = (-1,1) = \begin{bmatrix} -1 & -1 \\ 0 & 4 \end{bmatrix} \implies \lambda_1 = -1\ \&\ \lambda_2 = 4 \implies$ *Saddle point*

$A = \frac{\partial f}{\partial x}|x^* = (0,-1) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \implies \lambda_1 = 1\ \&\ \lambda_2 = 2 \implies$ *Unstable Node*

$A = \frac{\partial f}{\partial x}|x^* = (1,-1) = \begin{bmatrix} -1 & -1 \\ 0 & 4 \end{bmatrix} \implies \lambda_1 = -1\ \&\ \lambda_2 = 4 \implies$ *Saddle point*

b) For display purposes, it is plotted here by pplane in the following figure.



Solution3:

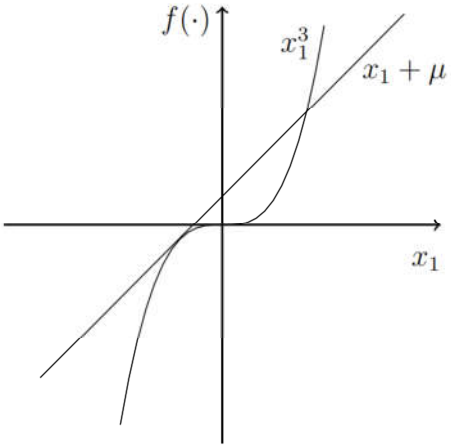
1- The equilibrium points are computed by

$$0 = \mu - x_{1_{eq}}\left(x_{1_{eq}}^2 - 1\right)$$

$$0 = -x_{2_{eq}}$$

From the second equation at any equilibrium point we should have $x_{2_{eq}} = 0$. From the first equation

$$x_{1_{eq}}^3 = x_{1_{eq}} + \mu$$



Equilibrium points are given by $x_{eq} = \left[x_{1_{eq}}, x_{2_{eq}}\right]^T$. Hence, depending on the value of μ there can be up to three equilibrium points.

2- The Jacobian is

$$J = \begin{bmatrix} -3x_1^2 + 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Linearization of the system about each equilibrium point is given by

$$\dot{x} = J|_{x_{eq}} x$$

J has two negative eigenvalues for $\left|x_{1_{eq}}\right| > \frac{\sqrt{3}}{3}$, hence the equilibrium points are stable $\forall \mu$.

J has one positive (and one negative) eigenvalue for $\left|x_{1_{eq}}\right| < \frac{\sqrt{3}}{3}$, hence the equilibrium points are saddle nodes $\forall \mu$.

For $x_{1_{eq}} = \pm \frac{\sqrt{3}}{3}$ linearization is inconclusive.

Solution4:

1- Suppose : $h(x,y)=\frac{1}{ln\sqrt{x^2+y^2}}$

$$\begin{cases}\dot{x}=-x-y\,h(x,y)\\ \dot{y}=-y+x\,h(x,y)\end{cases}$$

$$A=\frac{\partial f}{\partial x}|x=x^*=\begin{bmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{bmatrix}=\begin{bmatrix}-1-y\frac{\partial h(x,y)}{\partial x} & -h(x,y)-y\frac{\partial h(x,y)}{\partial y}\\ h(x,y)+x\frac{\partial h(x,y)}{\partial x} & -1+x\frac{\partial h(x,y)}{\partial y}\end{bmatrix}$$

$if\ \ \mathrm{x},\mathrm{y}\to 0\ \ \Longrightarrow h(x,y)\to 0$

then

$$A=\frac{\partial f}{\partial x}|x^*=(0,0)=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\ \Longrightarrow\ \lambda_1=-1\ \&\ \lambda_2=-1\Longrightarrow\ Stable\ node$$

Note that we have multiple eigenvalues, the equilibrium of the nonlinear system might be a node or a focus.

2- In order to draw the phase plane of the system, we use its polar relation (see slide 48 chapter 2):

$r\dot{r}=x\dot{x}+y\dot{y}$

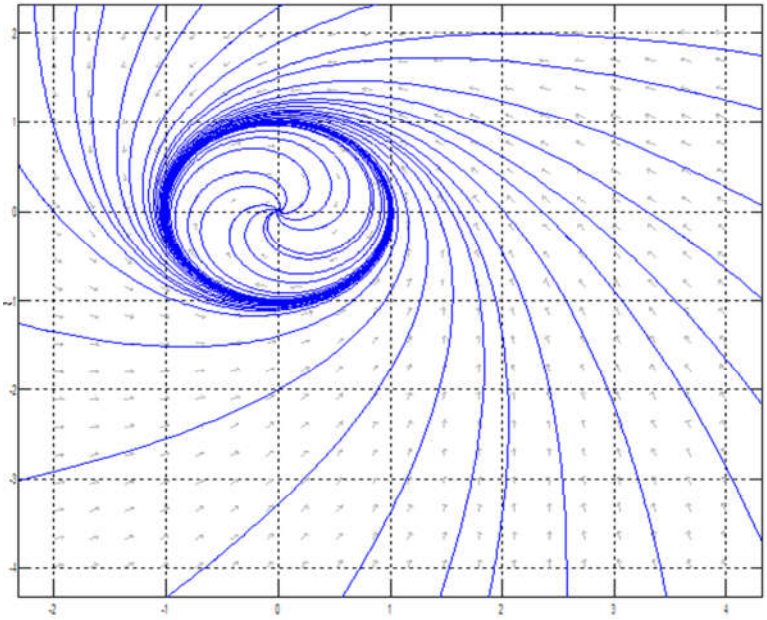
$r^2\dot{\theta}=x\dot{y}-\dot{x}y$

$r\dot{r}=-x^2-\frac{xy}{\ln r}-y^2+\frac{xy}{\ln r}=-r^2$

$r^2\dot{\theta}=-xy+\frac{x^2}{\ln r}+xy+\frac{x^2}{\ln r}=\frac{r^2}{\ln r}$

$$\Longrightarrow\ \begin{cases}\dot{r}=-r\\ \dot{\theta}=\frac{1}{\ln(r)}\end{cases}\ \Longrightarrow\ \begin{cases}r(t)=r_0e^{-t}\\ \theta(t)=\frac{1}{\ln(r_0)-t}+\theta_0\end{cases}$$

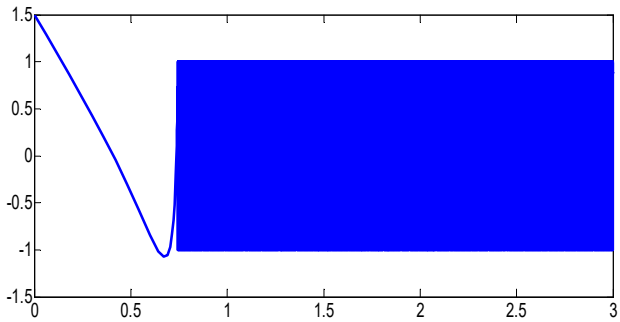
trajectories in the phase plane and the phase portrait resembles a $stable\ focus$.



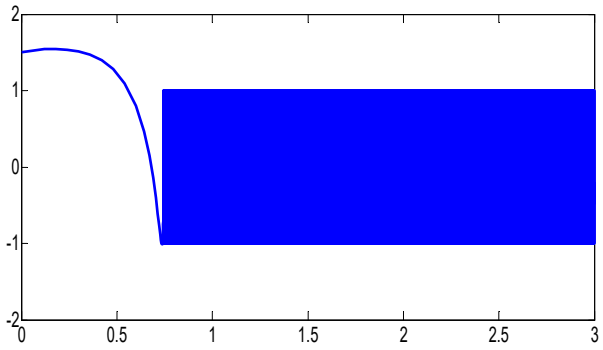
3) A stable focus next to cycle. Because it converges to the origin for the initial conditions inside the system cycle and for the initial conditions outside the system cycle to cycle.

Based on initial value [1.5,1.5]:

Graph of changes x in time

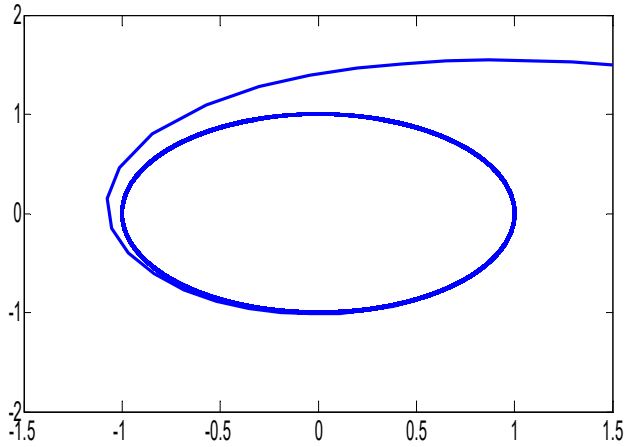


Graph of changes y in time



As it can be seen in both states after about 0.75s the frequency of oscillations tends to infinite.

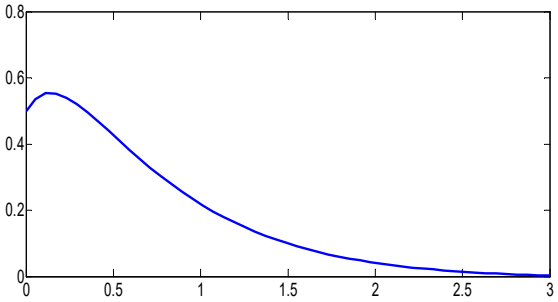
Graph the changes x1, x2 in a coordinates:



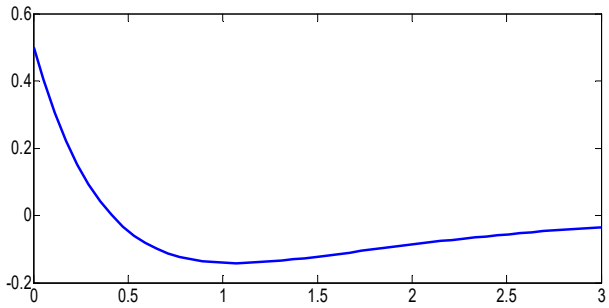
For the initial conditions $[1.5,1.5]$ it has converged to a cycle with a single radius.

Based on initial value $[0.5,0.5]$:

Graph of changes x in time

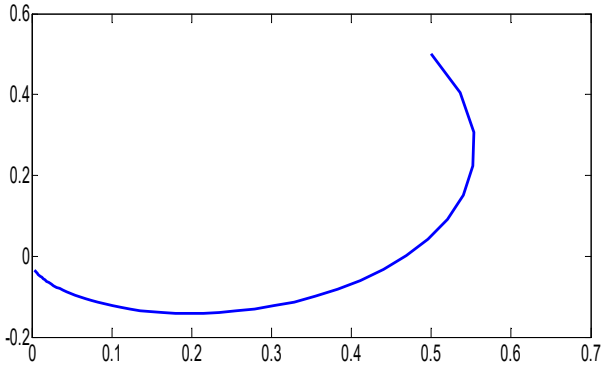


Graph of changes y in time



As can be seen, both system states have converged to zero.

Graph the changes x1, x2 in a coordinates



For initial conditions equal to $[0.5,0.5]$, the system converges to a stable focus $[0,0]$.

Solution5:

1- equilibria:

$$k_1 x_2 \left(1 - \frac{x_1}{1 + {x_2}^2}\right) = 0 \qquad (1)$$

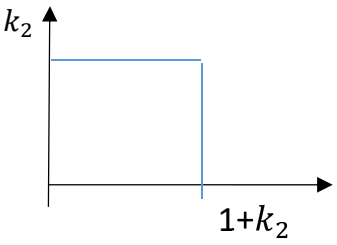
$$k_2 - x_2 - \frac{4x_1 x_2}{1 + {x_2}^2} = 0 \qquad (2)$$

From (1), $x_2 = 0$ or $\frac{x_1}{1+{x_2}^2} = 1$,but $x_2 = 0 \rightarrow k_2 = 0$ is contradiction

$$\frac{x_1}{1 + {x_2}^2} = 1 \rightarrow k_2 - x_2 - 4x_2 = 0 \rightarrow x_2 = \frac{k_2}{5}$$

So unique equilibrium $\rightarrow (1 + (\frac{k_2}{5})^2, \frac{k_2}{5})$

2- easy to show invariance of K:



On $x_1 = 0, \dot{x_1} = k_1 x_2 \geq 0$ if $x_2 \geq 0$

On $x_2 = 0, \dot{x_2} = k_2 > 0$

On $x_1 = 1 + {k_2}^2, \dot{x_1} = k_1 x_2 \left(1 - \frac{1+{k_2}^2}{1+{x_2}^2}\right) < 0$ if $\quad 0 < x_2 < k_2$

On $x_2 = k_2, \dot{x_2} = -\frac{4x_1 k_2}{1+{k_2}^2} < 0$ for $x_1 > 0$

K is invariant.

In order to have a closed orbit in K, we need that the only equilibrium point be unstable.

characteristic equation in equilibrium point:

$$s^2 + s\left(5 - 3\left(\left(\frac{k_2}{5}\right)^2 + \frac{k_2 k_1}{5}\right)\right) + k_2 k_1 = 0$$

From characteristic equation in equilibrium point:

$$5 - 3\left(\left(\frac{k_2}{5}\right)^2 + \frac{k_2 k_1}{5}\right) < 0$$

$$k_1 < 3\frac{k_2}{5} - \frac{25}{k_2}$$

3- when $k_1 > 3\frac{k_2}{5} - \frac{25}{k_2}$,equilibrium point is stable

when $k_1 < 3\frac{k_2}{5} - \frac{25}{k_2}$,equilibrium point is unstable, is surrounded by an orbit

so superficial Hopf bifurcation @ $k_1 = 3\frac{k_2}{5} - \frac{25}{k_2}$

Solution of The Extra Exercise:

1- $\dot{x} = ax \rightarrow x(t) = C_1 e^{\alpha t}$, where C_1 is a constant. The rabbit population will increase exponentially.

2- $\dot{y} = -\gamma y \rightarrow y(t) = -C_2 e^{\gamma t}$, where C_2 is a constant. The fox population will decrease exponentially.

3- Set:

$$\dot{x} = x(\alpha - \beta y) = 0$$

$$\dot{y} = -y(\gamma - \delta x) = 0$$

This system has two isolated equilibriums: at $(0,0)$ (which means an extinction of both species) and at $(\frac{\gamma}{\delta}, \frac{\alpha}{\beta})$. The last solution represents a fixed point at which both populations sustain their current, non-zero numbers, and, in the simplified model, do so indefinitely. The levels of population at which this equilibrium is achieved depend on the chosen values of the parameters.

We may calculate the Jacobian:

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{bmatrix}$$

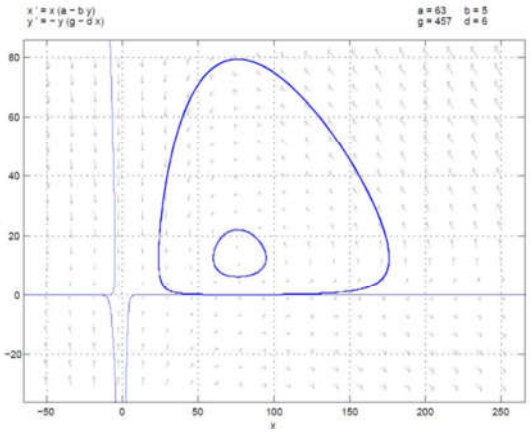
In the equilibrium points:

$$(0,0) \rightarrow \lambda_1 = \alpha_1, \lambda_1 = -\gamma$$

$$\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) \rightarrow \lambda_{1,2} = \pm i\sqrt{\gamma\alpha}$$

From the eigenvalues, we see that $(0, 0)$ is a saddle point and $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ is a center point.

4- See the phase portrait in below. A center point is lying in $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) \approx (76{:}\,2, 12{:}\,6)$ and a saddle point is lying in the origin, both as expected.



5- No, it is not possible to cross the lines $x = 0$ and $y = 0$, because they are invariant sets. If the states end up in these sets they will stay there forever, as we discussed in exercise (a)-(b).

6- Bendixson's criterion states that the system has no periodic orbits lying entirely in a simply connected region D if $\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}$ is not identically zero and does not change sign in D. It follows that such sets are given by:

$$D_1 = \{R^2 | \, \alpha - \beta y - (\gamma - \delta x) > 0\}$$

$$D_2 = \{R^2 | \, \alpha - \beta y - (\gamma - \delta x) < 0\}$$