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A Robust Linear Controller for Flexible Joint Manipulators

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***Abstract*— In this paper a new and completely linear algorithm is proposed for composite robust control of flexible joint robots. Moreover, the robust stability of the closed loop system in presence of structured and unstructured uncertainties is analyzed. To introduce the idea, flexible joint robot with structured and unstructured uncertainties is modelled and converted into singular perturbation form. A robust linear control algorithm is proposed for the slow dynamics and its robust stability conditions are derived using Thikhonov's theorem. Then the robust stability of the total system considering the porposed composite controller is analyzed, and sufficient conditions for robust stability of system is obtained. Finally the effectiveness of the proposed controller is verified through simulations. It is shown that not only the tracking performance of the proposed controller is very suitable, but also the actuator effort is much smaller than previous result.**

***Index Terms*— Flexible joint robots, harmonic drive, singular perturbation, robust PID, Thikhonov's theorem, UUB Stability, Lyapunov analysis, simulations.**

I. Introduction

Joint flexibility is one of the main reasons of robotic systems complexities. As it is shown [10] for practical applications, in order to achieve better tracking performance, joint flexibility must be taken in to account in both modeling and control.

The most important reason of joint flexibility is power transmision system flexibility. Mechanical arms need actuators capable of generating high torque in low speeds. On the contrary, electric motors provide robots with necessary torques only in high speeds. Therefore, many robots moved by by electric motors, use a power transmision system (gear box) for increasing torque and decreasing speed. among power transmision systems, harmonic drives have becomed more attractive to robot designers due to their special features. Unusual performance of harmonic drives theeth engagement give rise to achive high torque performance, high efficiency and $\cdots$ in a small volume [14]. However, joint flexibility and nonlinearity, in addition to increasing complexity of arms modeling, is a potential factor of system uncertainty that can affect favorable features of system and even in some case, leads to instability.

Due to existing joint flexibility, actuators' position (for example angle of motors' shaft) depends not directly to the driven arms' position. This is not favorable for applications with high precision. Moreover, unwanted oscillations due to joint flexibility, imposes bandwith limitation on all algorithms designed base on rigid robots and may create stability problems for feedback controls that neglect joint flexibility.

A large number of control schems have been proposed for compensating joint flexibility including singular perturbation theory [6], feedback linearization [10], adaptive control [3], [4], robust control [5], [12], [13] and intelligent methods [15]. Given slight joints flexibility, singular perturbation theory is applied as the base theory for modeling of robots. In this method, by applying Two-Time scale behavior, these systems are divided into slow and fast subsystem, and provides the initial means for corresponding control algorithms. As it is shown for a three axis model with flexible joints [1], such system are not always feedback linearizable. Thus usual methods such as computed torque can not be applied. By neglacting effect of axis movement on rotors kenimatic energy, a mathematicall model of these robots is derived in [10]. Considering this simplification system is feedback linearizable, although it should be noted that for appling this method, measurment of accelleration and jerk is required which is quite expensive.

In this paper, a linear control algorithm (PID with corrective term) is proposed the system. The robust stability of the closed loop system against structured and unstructured uncertainties is analyzed in detail. In order to develop this study, after a brief review of robot dynamics and PID control algorithm, an uncertain FJR with structuredand unstructured uncertainties is introduced in a singular perturbation form. Then with the aid of Thikhonov's theorem, by seperation of slow and fast variables, an algorithm for designing PID controller for corresponding rigid model and corrective term for flexible joints compensation is represented. Total stability of system is demonstrated then, and sufficient condition for its stability is presented. Simulations illustrate the effectiveness of the proposed control law, despite its very simple structure.

II. PID CONTROL OF RIGID ROBOT

Dynamic model of a n axis rigid robot is [2]:

$$M_t(q)\ddot{q} + N_t(q,\dot{q}) = u_0 \tag{1}$$

where,

$$\begin{cases} M_t(q) = M(q) + J \\ N_t(q,\dot{q}) = V_m(q,\dot{q})\dot{q} + G(q) + F_d\dot{q} + F_s(\dot{q}) + T_d \end{cases} \tag{2}$$

in which, $M(q)$ is $n \times n$ mass matrix,$V_m(q,\dot{q})$ is $n \times n$ matrix consisting of coriolis and centrifugal terms, $G(q)$ is $n \times 1$ vector of gravity terms, F_d is diagonal $n \times n$ matrix of Viscous friction, $F_s(\dot{q})$ is $n \times 1$ vector of Coulomb friction, T_d is $n \times 1$ vector of disturbance or unmodelled but bounded dynamics and J is $n \times n$ diagonal matrix of actuators masses. As it is demonstrated in [2], [9], in spite of uncertainty in all parameters, we have:

$$\begin{cases} \underline{m}_t I \leq M_t(q) \leq \overline{m}_t I \\ \|N_t\| \leq \beta_0 + \beta_1\|L\| + \beta_2\|L\|^2 \\ \|V_m\| \leq \beta_3 + \beta_4\|L\| \end{cases} \tag{3}$$

that $\underline{m}_t, \overline{m}_t, \beta_0, \beta_1.\beta_2, \beta_3, \beta_4$ are real positive constants and $L = [e^T \quad \dot{e}^T]$, $\|.\|$ is stands for eculudian norm. By choosing u_0 as

$$u_0 = K_V\dot{e} + K_P e + K_I\int_0^t e(s)ds = Kx \tag{4}$$

where,

$$\begin{cases} e = q_d - q \\ K = [K_I \quad K_P \quad K_V] \\ x = [\int_0^t e^T(s)ds \quad e^T \quad \dot{e}^T]^T \end{cases} \tag{5}$$

and by appling it in (1) we have:

$$\dot{x} = Ax + B\Delta A \tag{6}$$

in which,

$$A = \begin{bmatrix} \emptyset & I_n & \emptyset \\ \emptyset & \emptyset & I_n \\ -M_t^{-1}K_I & -M_t^{-1}K_P & -M_t^{-1}K_V \end{bmatrix} \tag{7}$$

$$B = \begin{bmatrix} \emptyset \\ \emptyset \\ M_t^{-1} \end{bmatrix}, \ \Delta A = N_t + M_t\ddot{q}_d \tag{8}$$

A. Choosing Lyapunov Function and Proof of Stability

We nominate a lyapunov function for the closed loop system as follows:

$$V(x) = x^T P x = \frac{1}{2}[\alpha_2\int_0^t e(s)ds + \alpha_1 e + \dot{e}]^T \cdot M_t \cdot [\alpha_2\int_0^t e(s)ds + \alpha_1 e + \dot{e}] + w^T P_1 w \tag{9}$$

where,

$$w = \begin{bmatrix} \int_0^t e(s)ds \\ e \end{bmatrix}, \ P_1 = \frac{1}{2}\begin{bmatrix} \alpha_2 K_P + \alpha_1 K_I & \alpha_2 K_V + K_I \\ \alpha_2 K_V + K_I & \alpha_1 K_V + K_P \end{bmatrix} \tag{10}$$

and so

$$P = \frac{1}{2}\begin{bmatrix} \alpha_2 K_P + \alpha_1 K_I + \alpha_2^2 M_t & \alpha_2 K_V + K_I + \alpha_1\alpha_2 M_t & \alpha_2 M_t \\ \alpha_2 K_V + K_I + \alpha_1\alpha_2 M_t & \alpha_1 K_V + K_P + \alpha_1^2 M_t & \alpha_1 M_t \\ \alpha_2 M_t & \alpha_1 M_t & M_t \end{bmatrix}$$

Since M_t is a poisitive definite matrix, P is positive definite if and only if P_1 is positive definite matrix. if

$$\begin{cases} K_P = k_P I \\ K_V = k_V I \\ K_I = k_I I \end{cases} \tag{11}$$

then, the following lemma shows that by proper choose of controller coefficients, P becomes positive definite with lower and upper bounds.

Lemma(1): we suppose that following inequalities are true:

$$\alpha_1 > 0 \quad \alpha_2 > 0 \quad \alpha_1 + \alpha_2 < 1$$
$$s_1 = \alpha_2(k_P - k_V) - (1-\alpha_1)k_I - \alpha_2(1+\alpha_1-\alpha_2)\overline{m}_t > 0$$
$$s_2 = k_P + (\alpha_1 - \alpha_2)k_V - k_I - \alpha_1(1+\alpha_2-\alpha_1)\overline{m}_t > 0$$

then P is a positive definite and satisfies this condition (Rayleigh-Ritz) [7]:

$$\underline{\lambda}(P)\|x\|^2 \leq V(x) \leq \overline{\lambda}(P)\|x\|^2 \tag{12}$$

where,

$$\underline{\lambda}(P) = Min\{\frac{1-\alpha_1-\alpha_2}{2}\underline{m}_t, \frac{s_1}{2}, \frac{s_2}{2}\}$$
$$\overline{\lambda}(P) = Max\{\frac{1+\alpha_1+\alpha_2}{2}\overline{m}_t, \frac{s_3}{2}, \frac{s_4}{2}\}$$

and

$$s_3 = \alpha_2(k_P + k_v) + (1+\alpha_1)k_I + (1+\alpha_1+\alpha_2)\alpha_2\overline{m}_t$$
$$s_4 = \alpha_1\overline{m}_t(1+\alpha_1+\alpha_2) + (\alpha_1+\alpha_2)k_V + k_P + k_I$$

proof of this lemma is like that in [9], by appling gershgorian's theorem. By proving the fact that P is positive definite :

$$\dot{V}(x) = x^T(A^TP + PA + \dot{P})x + 2x^TPB\Delta A$$
$$= -x^TQx + \tfrac{1}{2}x^T\begin{bmatrix} \alpha_2 I \\ \alpha_1 I \\ I \end{bmatrix}\dot{M}_t[\alpha_2 I \quad \alpha_1 I \quad I]x + \tfrac{1}{2}x^T\begin{bmatrix} \emptyset & \alpha_2^2 I & \alpha_1\alpha_2 I \\ \alpha_2 I & 2\alpha_1\alpha_2 I & (\alpha_1^2+\alpha_2)I \\ \alpha_1\alpha_2 I & (\alpha_1^2+\alpha_2)I & \alpha_1 I \end{bmatrix}\cdot \begin{bmatrix} M_t & \emptyset & \emptyset \\ \emptyset & M_t & \emptyset \\ \emptyset & \emptyset & M_t \end{bmatrix}x + x^T\begin{bmatrix} \alpha_2 I \\ \alpha_1 I \\ I \end{bmatrix}\Delta A$$

since we have [8]:

$$y^T\dot{M}_t y = 2y^TV_m y \tag{13}$$

so

$$\dot{V}(x) = -x^TQx + \frac{1}{2}x^T\begin{bmatrix} \alpha_2 I \\ \alpha_1 I \\ I \end{bmatrix}(V_m + V_m^T)[\alpha_2 I \quad \alpha_1 I \quad I]\cdot x + \tfrac{1}{2}x^T\begin{bmatrix} \emptyset & \alpha_2^2 I & \alpha_1\alpha_2 I \\ \alpha_2 I & 2\alpha_1\alpha_2 I & (\alpha_1^2+\alpha_2)I \\ \alpha_1\alpha_2 I & (\alpha_1^2+\alpha_2)I & \alpha_1 I \end{bmatrix}\cdot \begin{bmatrix} M_t & \emptyset & \emptyset \\ \emptyset & M_t & \emptyset \\ \emptyset & \emptyset & M_t \end{bmatrix}x + x^T\begin{bmatrix} \alpha_2 I \\ \alpha_1 I \\ I \end{bmatrix}\Delta A$$

in which:

$$Q = \begin{bmatrix} \alpha_2 k_I I & \emptyset & \emptyset \\ \emptyset & (\alpha_1 k_P - \alpha_2 k_V - k_I)I & \emptyset \\ \emptyset & \emptyset & k_V I \end{bmatrix} \tag{14}$$

so we have:

$$\dot V(x) \le -\gamma\|x\|^2 + \lambda_1\|V_m\|\|x\|^2 + \lambda_2\overline{m}_t\|x\|^2 + \alpha_2^{-1}\lambda_1\|x\|\|\Delta A\| \quad (15)$$

$$\Longrightarrow \dot V(x) \le \|x\|(\xi_0 - \xi_1\|x\| + \xi_2\|x\|^2) \quad (16)$$

$$\gamma = Min\{\alpha_2 k_I, \alpha_1 k_P - \alpha_2 k_V - k_I, k_V\} \quad (17)$$

Now according to (3), (8), (15), (16) and $\|L\| \le \|x\|$ we have:

$$\xi_0 = \alpha_2^{-1}\lambda_1\beta_0 + \alpha_2^{-1}\lambda_1\lambda_3\overline{m}_t \quad (18)$$

$$\xi_1 = \gamma - \lambda_1\beta_3 - \lambda_2\overline{m}_t - \alpha_2^{-1}\lambda_1\beta_1 \quad (19)$$

$$\xi_2 = \lambda_1\beta_4 + \alpha_2^{-1}\lambda_1\beta_2 \quad (20)$$

in which: $\lambda_1 = \lambda_{Max}(R_1)$, $\lambda_2 = \lambda_{Max}(R_2)$, $\lambda_3 = sup\|\ddot q_d\|$ and $\lambda_{Min}, \lambda_{Max}$ are the greatest and smallest eigen values respectively and

$$R_1 = \begin{bmatrix} \alpha_2^2 I & \alpha_1\alpha_2 I & \alpha_2 I \\ \alpha_1\alpha_2 I & \alpha_1^2 I & \alpha_1 I \\ \alpha_2 I & \alpha_1 I & I \end{bmatrix}$$

$$R_2 = \frac{1}{2}\begin{bmatrix} \emptyset & \alpha_2^2 I & \alpha_1\alpha_2 I \\ \alpha_2 I & 2\alpha_1\alpha_2 I & (\alpha_1^2+\alpha_2)I \\ \alpha_1\alpha_2 I & (\alpha_1^2+\alpha_2)I & \alpha_1 I \end{bmatrix}$$

According to the derived results, following theorem which demonestrats the UUB stability of the error system (6), can be stated:

Theorem 1: Error system (6) is UUB stable if ξ_1 is choosen sufficiently large.

Proof: According to (12), (16) and lemma 3.5 of [8], if these conditions are met, the system is UUB stable against B(o,d), that

$$d = \frac{2\xi_0}{\xi_1 + \sqrt{\xi_1^2 - 4\xi_0\xi_2}}\sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)}} \quad (21)$$

and

$$\xi_1 > 2\sqrt{\xi_0\xi_2} \quad (22)$$

$$\xi_1^2 + \xi_1\sqrt{\xi_1^2 - 4\xi_0\xi_2} > 2\xi_0\xi_2(1 + \sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)}}) \quad (23)$$

$$\xi_1 + \sqrt{\xi_1^2 - 4\xi_0\xi_2} > 2\xi_2\|x_0\|\sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)}} \quad (24)$$

By enlarging ξ, these conditions are met easily and ξ is enlarged by raising K_P , K_V , K_I. ■

III. Flexible Joints Robot

Practical results from industrial robots using harmonic drive as power transmision system show the great impact of joints flexibility on systems dynamic. By assuming that joint rigidity is large with respect to other systems parameters and damping factor of joint is small, dynamic model of n axis flexible joint robot can be written as follow [3]:

$$\begin{cases} M(q_1)\ddot q_1 + N(q_1,\dot q_1) = K(q_2 - q_1) \\ J\ddot q_2 = K(q_1 - q_2) + u \end{cases} \quad (25)$$

$$N(q_1,\dot q_1) = V_m(q_1,\dot q_1)\dot q_1 + G(q_1) + F_d\dot q_1 + F_s(\dot q_1) + T_d \quad (26)$$

q_1 and q_2 are angles of shaft and motor and K is $n \times n$ diagonal matrix representing joints stiffness. By assuming that all joints stiffness are the same (This assumption does not reduce the generality of problem, for the general case can be easily reached by variable scaling) and since it is assumed to be large with respect to other systems parameters, it can be reduced in the form of $O(1/\epsilon^2)$. Notice that by assuming the presence of uncertainty when all joints are rotational, we have [2], [8]:

$$\begin{cases} m_1 I \le M(q_1) \le m_2 I \\ ||V_m(q_1,\dot q_1|| \le \zeta_c||\dot q_1|| \\ ||G(q_1)|| \le \zeta_g \end{cases} \quad (27)$$

and

$$\begin{cases} ||F_d\dot q_1 + F_s(\dot q_1)|| = \zeta_{f0} + \zeta_{f_1}||q_1|| \\ j_1 I \le J \le j_2 I \end{cases} \quad (28)$$

By assuming the disturbance to be limited we have:

$$||T_d|| \le \zeta_e \quad (29)$$

im which $\zeta_e, j_2, j_1, \zeta_{f1}, \zeta_{f0}, \zeta_g, \zeta_c, m_2, m_1$ are real positive constants. In case that all joints are rigid, then the model of system reduses to:

$$M_t(q)\ddot q + N_t(q,\dot q) = u_r \quad (30)$$

in which $q = q_1$, M_t is a positive definite mass matrix. This model is a special case of FJR model when $K \to \infty$.
This fact is due to that flexible joint model is a singular perturbation model which can be extracted from rigid model. [11].

A. Control

In this section, the modification required to use the rigid model control low (4) for flexible joint manipulators. First consider adding a corrective term to the the control law in the form of

$$u = u_r + K_d(\dot q_1 - \dot q_2) \quad (31)$$

Here u_r is the same PID controller given by (4) and K_d is a constant diagonal matrix whose elements are of order of $O(1/\epsilon)$.
By appling control low (31) in equation (25) and defining variable z as:

$$z = K(q_2 - q_1) \quad (32)$$

the closed loop dynamic equation reduces to:

$$J\ddot z + K_d\dot z + Kz = K(u_r - J\ddot q_1) \quad (33)$$

Considering the assumptions on K and choosing K_d of order $O(1/\epsilon)$:

$$K = \frac{K_1}{\epsilon^2} \quad ; \quad K_d = \frac{K_2}{\epsilon} \quad (34)$$

here K_1 and K_2 are of order $O(1)$. Equation (33) can be restated in following form:

$$\epsilon^2 J\ddot z + \epsilon K_2 z + K_1 z = K_1(u_r - J\ddot q_1) \quad (35)$$

Now equation (25) can be stated as follow:

$$\begin{cases} M(q_1)\ddot q_1 + N(q_1,\dot q_1) = z \\ \epsilon^2 J\ddot z + \epsilon K_2 z + K_1 z = K_1(u_r - J\ddot q_1) \end{cases} \quad (36)$$

System (36) is a singularly perturbed system, whose slow variables q_1 and $\dot q_1$ are shaft parameters and z and $\dot z$ are fast parameters. From the results of singular perturbation theory, flexible system (36) can be approximated by two quasi-steady state system and boundary layer system. Considering $\epsilon = 0$, for equation (36) we have:

$$\bar{z} = \bar{u}_r - J\ddot{\bar{q}}_1 \tag{37}$$

In which the variables with $(\bar{\cdot})$ are defined when $\epsilon = 0$. The dynamic equation is then reduced to:

$$(M(\bar{q}_1) + J)\ddot{\bar{q}}_1 + N(\bar{q}_1, \dot{\bar{q}}_1) = \bar{u}_r \tag{38}$$

This equation resembles the rigid robot model with $\bar{q}_1$, is known as quasi-steady state system. To use Thikhonov's theorem [6], notice that elastic force of joints z(t) and shaft angle q(t) for $t > 0$,nmeet these conditions:

$$\begin{cases} z(t) = \bar{z}(t) + \eta(\tau) + O(\epsilon) \\ q_1(t) = \bar{q}_1(t) + O(\epsilon) \end{cases} \tag{39}$$

in which $\tau = t/\epsilon$, is fast scale time and η is the the boundary layer fast state variable:

$$J\frac{d^2\eta}{d\tau^2} + K_2\frac{d\eta}{d\tau} + K_1\eta = 0 \tag{40}$$

Considering these results, flexible joint systems can be approximated up to order of $O(\epsilon)$ as follows:

$$\begin{cases} (M(q_1) + J)\ddot{q}_1 + N(q_1, \dot{q}_1) = u_r + \eta(t/\epsilon) \\ J\frac{d^2\eta}{d\tau^2} + K_2\frac{d\eta}{d\tau} + K_1\eta = 0 \end{cases} \tag{41}$$

Hence, K_2 can be chosen suitably such that boundary layer system (40) becomes asymptatically stable. Therefore, with sufficiently small values of ϵ, the composite control consisting of the rigid control u_r (PID) and the corrective term $K_d(\dot{q}_1 - \dot{q}_2)$, resembles the flexible joint robot response to that of rigid system controlled by only u_r, after some initially damped transient of fast variables $\eta(t)$.

IV. Analysis of Robust Stability of Total System

PID control of rigid model and its stability is discussed in previous section, also it is demonstrated that boundary layer system is asymptatically stable, due to the corrective term. As it is known in general, seperate stability of boundary layer and quasi-steady state subsystems does not guarantee total system stability [6]. In this section total system UUB stability is analyzed, based on the results of previous section. To begin, reconsider the governing dynamic equations of FJR:

$$\begin{cases} (M(q_1) + J)\ddot{q}_1 + N(q_1, \dot{q}_1) = u_r + \eta(t/\epsilon) \\ J\frac{d^2\eta}{d\tau^2} + K_2\frac{d\eta}{d\tau} + K_1\eta = 0 \end{cases} \tag{42}$$

By placing u_r from (4) and noticing that $e = q_d - q_1$:

$$\begin{bmatrix} e \\ \dot{e} \\ \ddot{e} \end{bmatrix} = \begin{bmatrix} \emptyset & I & \emptyset \\ \emptyset & \emptyset & I \\ -M_t^{-1}K_I & -M_t^{-1}K_P & -M_t^{-1}K_V \end{bmatrix} \begin{bmatrix} \int_0^t e(s)ds \\ e \\ \dot{e} \end{bmatrix} + \begin{bmatrix} \emptyset \\ \emptyset \\ M_t^{-1} \end{bmatrix} (N_t + M_t\ddot{q}_d) + \begin{bmatrix} \emptyset \\ \emptyset \\ -M_t^{-1} \end{bmatrix} \eta$$

Hence,

$$\begin{bmatrix} \dot{\eta} \\ \ddot{\eta} \end{bmatrix} = \begin{bmatrix} \emptyset & I \\ -J^{-1}K & -J^{-1}K_d \end{bmatrix} \begin{bmatrix} \eta \\ \dot{\eta} \end{bmatrix} \tag{43}$$

since $x = [\, \int_0^t e(s)^T ds \quad e^T \quad \dot{e}^T \,]^T$, $y = [\, \eta^T \quad \dot{\eta}^T \,]^T$ we have:

$$\dot{x} = Ax + B\Delta A + C[\, I \quad \emptyset \,]y \tag{44}$$

$$\dot{y} = \tilde{A}y \tag{45}$$

in which,

$$A = \begin{bmatrix} \emptyset & I & \emptyset \\ \emptyset & \emptyset & I \\ -M_t^{-1}K_I & -M_t^{-1}K_P & -M_t^{-1}K_V \end{bmatrix} \tag{46}$$

$$\Delta A = N_t + M_t\ddot{q}_d \quad , \quad B = \begin{bmatrix} \emptyset \\ \emptyset \\ M_t^{-1} \end{bmatrix} \tag{47}$$

$$C = \begin{bmatrix} \emptyset \\ \emptyset \\ -M_t^{-1} \end{bmatrix} \quad , \quad \tilde{A} = \begin{bmatrix} \emptyset & I \\ -J^{-1}K & -J^{-1}K_d \end{bmatrix} \tag{48}$$

Theorem 2: There is a positive definite matrix K_d such that closed-loop system described with (45) is asymptotically stable.

Proof: The lyapunov function candidate is as follow:

$$V_F = y^TSy \quad , \quad S = \frac{1}{2}\begin{bmatrix} K_d + K & J \\ J & J \end{bmatrix} \tag{49}$$

In order for S to be positive definite, it is sufficient that $K_d > J$, now by differentiation of V_F along system trajectory (45), we have:

$$\dot{V}_F = \dot{y}^TSy + y^TS\dot{y} = -\eta^TK\eta - \dot{\eta}^T(K_d - J)\dot{\eta} < 0 \tag{50}$$

since K, K_d, J are diagonal positive definite matrices, hence, $\dot{V}_F$ becomes negative defenite.

$$\dot{V}_F = -y^TWy \quad , \quad W = \begin{bmatrix} K & \emptyset \\ \emptyset & K_d - J \end{bmatrix} \tag{51}$$

■

Theorem 3: Closed loop systems (44) and (45) is UUB stable if K_d and ξ_1 are chosen sufficiently large.

Proof: Consider the following lyapunov function candidate:

$$V(x, y) = x^TPx + y^TSy \tag{52}$$

x^TPx is the considered as the lyapunov function for rigid system and y^TSy is that in theorem (2). According to Rayleigh-Ritz inequality:

$$\begin{cases} \underline{\lambda}(P)\|x\|^2 \le x^TPx \le \overline{\lambda}(P)\|x\|^2 \\ \underline{\lambda}(S)\|y\|^2 \le y^TSy \le \overline{\lambda}(S)\|y\|^2 \end{cases} \tag{53}$$

in which $\underline{\lambda}, \overline{\lambda}$ are the smallest and largest eignvalues, respectively. By adding these inequalities:

$$\underline{\lambda}(S)\|y\|^2 + \underline{\lambda}(P)\|x\|^2 \le V(x, y) \le \overline{\lambda}(S)\|y\|^2 + \overline{\lambda}(P)\|x\|^2 \tag{54}$$

Defining

$$Z_t = [\, \|x\| \quad \|y\| \,]^T \tag{55}$$

we have:

$$[\, \|x\| \quad \|y\| \,] \begin{bmatrix} \underline{\lambda}(P) & 0 \\ 0 & \underline{\lambda}(S) \end{bmatrix} \begin{bmatrix} \|x\| \\ \|y\| \end{bmatrix} \le V(x, y)$$

$$\leq [\,\|x\| \quad \|y\|\,] \begin{bmatrix} \overline{\lambda}(P) & 0 \\ 0 & \overline{\lambda}(S) \end{bmatrix} \begin{bmatrix} \|x\| \\ \|y\| \end{bmatrix} \tag{56}$$

By appling Rayleigh-Ritz inequality again, we have:

$$\underline{\lambda}\|Z_t\| \leq V(Z_t) \leq \overline{\lambda}\|Z_t\| \tag{57}$$

in which

$$\underline{\lambda} = Min\{\underline{\lambda}(P), \underline{\lambda}(S)\} \tag{58}$$
$$\overline{\lambda} = Max\{\overline{\lambda}(P), \overline{\lambda}(S)\} \tag{59}$$

Now by deriving from (52) in direction of (44) and (45) we have :

$$\begin{aligned} \dot{V} &= 2x^T P\dot{x} + x^T \dot{P}x + 2y^T S\dot{y} \\ &= [2x^T P(Ax + B\Delta A) + x^T \dot{P}x] \\ &\quad +2x^T PC[I \quad \emptyset]y + 2y^T S\dot{y} \end{aligned} \tag{60}$$

According to (16) it can be concluded that:

$$2x^T P(Ax + B\Delta A) + x^T \dot{P}x \leq \|x\|(\xi_0 - \xi_1\|x\| + \xi_2\|x\|^2) \tag{61}$$

and also by defining $\gamma_1 = \lambda_{max}(M_t)$ we have:

$$2x^T PC[I \quad \emptyset]y \leq 2\gamma_1\overline{\lambda}(P)\|x\|\|y\| \tag{62}$$

as we saw in theorem (2):

$$2y^T S\dot{y} \leq -\lambda_{min}(W)\|y\|^2 \tag{63}$$

Hence,

$$\dot{V} \leq [\,\|x\| \quad \|y\|\,] \begin{bmatrix} \xi_1 & -\gamma_1\overline{\lambda}(P) \\ -\gamma_1\overline{\lambda}(P) & \lambda_{min}(W) \end{bmatrix} \begin{bmatrix} \|x\| \\ \|y\| \end{bmatrix} + \xi_0\|x\| + \xi_2\|x\|^3 \tag{64}$$

According to (55) we have:

$$\dot{V} \leq -Z_t^T RZ_t + \xi_0\|Z_t\| + \xi_2\|Z_t\|^3 \tag{65}$$

in which

$$R = \begin{bmatrix} \xi_1 & -\gamma_1\overline{\lambda}(P) \\ -\gamma_1\overline{\lambda}(P) & \lambda_{min}(W) \end{bmatrix} \tag{66}$$

For R to be positive definite it is required that

$$\lambda_{min}(W) > \frac{\gamma_1^2\overline{\lambda}^2(P)}{\xi_1} \tag{67}$$

By meeting condition (67) accomplished by suitable choise of K_d for fast subsystem, we have:

$$\dot{V} \leq \|Z_t\|(\xi_0 - \lambda_{min}(R)\|Z_t\| + \xi_2\|Z_t\|^2) \tag{68}$$

Now according to (57), (68) and lemma 3-5 from [8], system is UUB stable against $Y(0, d')$, in which:

$$d' = \frac{2\xi_0}{\lambda_{min}(R) + \sqrt{\lambda_{min}^2(R) - 4\xi_0\xi_2}}\sqrt{\frac{\overline{\lambda}}{\underline{\lambda}}} \tag{69}$$

if these conditions are met:

$$\lambda_{min}(R) > 2\sqrt{\xi_0\xi_2}$$

$$\lambda_{min}^2(R) + \lambda_{min}(R)\sqrt{\lambda_{min}^2(R) - 4\xi_0\xi_2} > 2\xi_0\xi_2(1 + \sqrt{\frac{\overline{\lambda}}{\underline{\lambda}}})$$

$$\lambda_{min}(R) + \sqrt{\lambda_{min}^2(R) - 4\xi_0\xi_2} > 2\xi_2\|Z_{t0}\|\sqrt{\frac{\overline{\lambda}}{\underline{\lambda}}}$$

TABLE I
ARM PARAMETERS (ALL UNITS ARE IN SI)

Parameters	Nominal Values
Mass	$M = 2$
Joint stifness	$k = 100$
Length(2L)	$L = 1$
Gravity coeff.	$g = 9.8$
Inertia	$I = 1.5$
Motor inertia	$J = 1.5$

These conditions are simply met by increasing $\lambda_{min}(R)$, through appropriate choice of large ξ_1, and $\lambda_{min}(W)$. ξ_1 is a function of the robust PID gains K_p, K_I and K_V, and $\lambda_{min}(W)$ are affected by the corrective term gains K_d. Therefore, by the choice of the controller gains such that the above conditions are met the robust stability of the closed-loop system is guaranteed. ∎

V. SIMULATIONS

For demonestrating effectiveness of the proposed control algorithm, the simulation of the closed loop single axis arm with flexible joint is performed. Governing equation of motion of this system are as follows [10]:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-MgL}{I}\sin(x_1) - \frac{k}{I}(x_1 - x_3) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{k}{J}(x_1 - x_3) + \frac{1}{J}u \end{aligned}$$

in which $x_3 = q_2$, $x_1 = q_1$, by choosing: $z = k(q_1 - q_2)$, $q_1 = q$, the equations of motion of system stated in the form of singular perturbation are:

$$\begin{aligned} \ddot{q} &= \frac{-MgL}{I}\sin(q) - \frac{1}{I}z \\ \epsilon\ddot{z} &= \frac{-MgL}{I}\sin(q) - (\frac{1}{I} + \frac{1}{J})z - \frac{1}{J}u \end{aligned}$$

The composite controller u is designed in the form of:

$$u = u_s + K_d(\dot{q}_1 - \dot{q}_2) \tag{70}$$

in which,

$$\begin{aligned} u_s &= 50\dot{e} + 60e + 30\int_0^t e(s)ds \\ K_d &= 50 \end{aligned}$$

It can be shown that the controller gains are sufficiently high enough to satisfy the theorem's stability conditions. For sake of comparison in the simulations, assume the desired trajectory in the form of:

$$\theta = 1.57 + 7.8539\exp(-t) - 9.428\exp(-t/1.2) \tag{71}$$

In this curve, joint angle reaches final value of $\theta = \pi/2$ from initial value of $\theta = 0$ with a soft transient. By applying rigid control u_s, system becomes unstable. The main reason for rigid controller instability is ignoring flexibility effects in the system. But by applying the proposed algorithm, system remains stable possesing a good tracking as it is seen in figure (1). Figure (2) illustrates the output of system with the composite controller proposed in [12] with the same

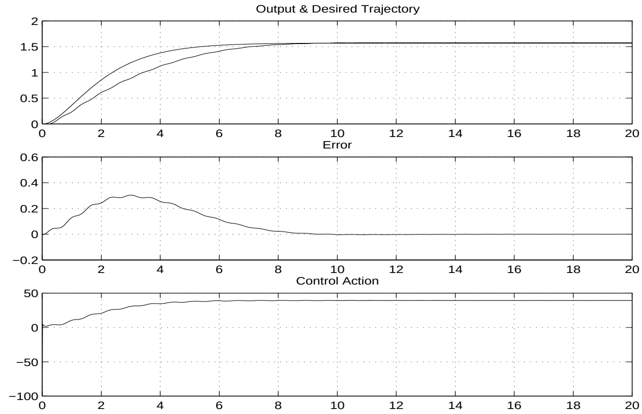

Fig. 1. Suitable tracking performance of the close loop system to a smooth reference trajectory; Proposed algorithm.

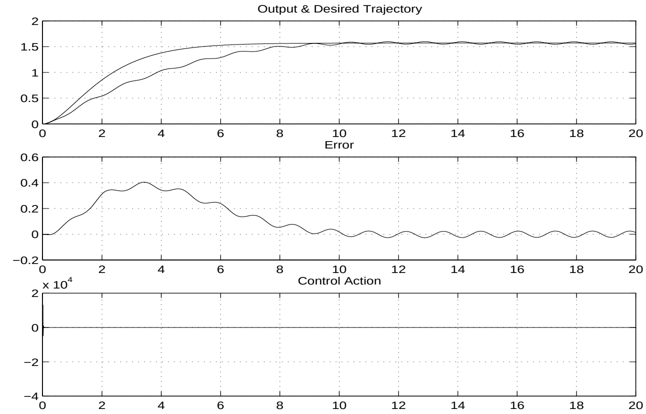

Fig. 2. Tracking performance of the close loop system; Taghirad et al algorithm.

PID control coefficients. This controller consists of three parts, and an integral manifold first corrective term with order approximation is considered to preserve the system total stability. Despite the simple structure of our proposed controller, the tracking performance is even better, and the actuator effort is much less than that in [12] (Figure (2)).
In order to verify the reachable bandwith for the system, $\phi = sin(4t)$ is considered as the next reference trajectory of the closed loop system. As it is illustrated in figure (3), the tracking performance is very well, with a suitabe control effort. In this case also the controller effort is much less than that reported in [12].

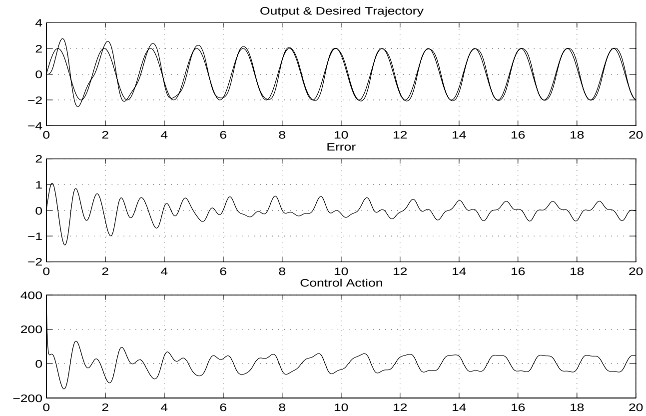

Fig. 3. Suitable tracking performance of the close loop system to a sinusoid trajectory; Proposed algorithm.

VI. Conclusions

In this paper a new algorithm is proposed for robust composite control of flexible joint robots. Despite similar natare of the proposed controller to that reported in [12] by authors, this controller is very simpler in structure and completely linear. It consists of only two terms, stabilizing the fast dynamics, and shaping the tracking performance with slow subsystem controller. To develop the idea, first flexible joint robot encapsulating its uncertainties is modelled and converted into singular perturbation form. The robust stability of the closed loop system is analyzed using Thikhonov's theorem. It is shown that similar to the controllers reported in [12] by authors, and despite the simpler structure proposed in here, the total closd loop system is UUB stable, only if, the controller gains are selected higher than a critical limit. The effectiveness of the proposed controller is verified through simulations, and the tracking performance is compared to the previous results. It is shown that not only the tracking performance of the proposed controller is very suitable, the actuator effort is much smaller than previous result. Finally, this simple structure controller, having its stability guranteed through the theorems, seems to be promising for further future development.

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