Advanced Robotics, Vol. 20, No. 2, pp. 181–211 (2006)

Also available online - www.vsppub.com

Full paper

Stability analysis and robust composite controller synthesis for flexible joint robots

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Received 16 November 2004; accepted 7 March 2005

Abstract—In this paper the control of flexible joint manipulators is studied in detail. The model of N-axis flexible joint manipulators is derived and reformulated in the form of singular perturbation theory and an integral manifold is used to separate fast dynamics from slow dynamics. A composite control algorithm is proposed for the flexible joint robots, which consists of two main parts. Fast control, u_{f}, guarantees that the fast dynamics remains asymptotically stable and the corresponding integral manifold remains invariant. Slow control, u_{s}, consists of a robust PID designed based on the rigid model and a corrective term designed based on the reduced flexible model. The stability of the fast dynamics and robust stability of the PID scheme are analyzed separately, and finally, the closed-loop system is proved to be uniformly ultimately bounded (UUB) stable by Lyapunov stability analysis. Finally, the effectiveness of the proposed control law is verified through simulations. The simulation results of single- and two-link flexible joint manipulators are compared with the literature. It is shown that the proposed control law ensures robust stability and performance despite the modeling uncertainties.

Keywords: Flexible joint robots; integral manifold; UUB stability; Lyapunov analysis; two-link flexible manipulator; performance stimulations; uncertainty.

1. INTRODUCTION

Multiple-axis robot manipulators are widely used in industrial and space applications. The high accuracy of these robots is due to their rigidity, which makes them highly controllable. After the inception of harmonic drive in 1955, and its wide acceptance and use in the design of many electrically driven robots, the rigidity of robot manipulators was greatly affected. In the early 1980s researchers showed that the use of control algorithms developed based on rigid robot dynamics on real non-rigid robots is very limited and may even cause instability [1]. The singular

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perturbation theory is used as the basic theory to model the dynamics of flexible joint robots (FJRs), in which, using two-time scale behavior, these systems are divided into fast and slow subsystems [2]. As shown in Ref. [3] for a three-axis flexible robot the system is not feedback linearizable and the use of methods such as computed torque methods for flexible manipulators is not directly implementable. By neglecting the effects of link motion on the kinetic energy of the rotor, Spong derived a mathematical model for such systems in which the system is feedback linearizable [4]. However, to linearize the system, acceleration and jerk feedback is required, whose measurement is very costly. To avoid the need for acceleration and jerk in this method the idea of an integral manifold is employed. In this method, instead of using the zero-order approximation of the model extracted from the singular perturbation theory, higher-order models can be used and, hence, a series of corrective terms is added to the control algorithm [1, 5]. In adaptive methods many algorithms are developed for FJRs, in most of which a term due to the fast subsystem is added to the adaptive algorithm based on rigid models [5]. In robust methods considering model uncertainties the stability of the fast subsystem is first analyzed and, by the use of robust control synthesis, a robust controller is designed for the slow subsystem [6, 7]. Moreover, some controllers have recently been designed for FJRs using the compliance control schemes [8, 9].

As has been shown, most research on FJRs has concentrated on non-linear control schemes. Using the singular perturbation and integral manifold concepts for modeling combined with the composite control approach, and using two distinct terms for slow and fast variables seems to be highly effective. However, since in this method the global stability is not guaranteed, many efforts towards stability analysis have been reported in the literature [10–12]. Qu has proposed a robust controller with local stability in Ref. [13], as a generalization of his previous ideas about rigid robots. Later, in Ref. [14], a robust controller has been proposed with guaranteed uniformly ultimately bounded (UUB) tracking in the presence of small disturbances and parametric uncertainty.

In this paper we propose a method based on a composite control structure and thoroughly analyze the robust stability of the overall uncertain system. In this analysis, similar to Refs [15, 16] the singular perturbation model of the FJR is used, but in the presence of the modeling uncertainties, and the system is divided into slow and fast subsystems. Then, an integral manifold in addition to a composite control law is introduced in order to retain the integral manifold invariant and to satisfy the asymptotic stability requirement. The control effort consists of three elements — the first element is designed for the fast subsystem, the second term is a robust PID control designed for the rigid subsystem and the third term is a corrective law designed based on the first-order approximation of the reduced flexible system. Moreover, unlike the stability analysis given in previous research, which was limited to the rigid model subsystem [16], in this paper the overall stability of the closed-loop system is thoroughly analyzed-based on Lyapunov stability theory. Stability conditions for the robust PID controller are derived to guarantee that the overall closed-loop system is UUB stable. In order to verify the effectiveness of the proposed control law and compare its performance to other methods given in the literature, simulations of single- and two-link flexible joint manipulators are examined. It is shown in this study that the proposed control law ensures robust stability and performance, despite the modeling uncertainties.

2. FJR MODELING

Spong [4] has derived a non-linear dynamic model for FJR using singular perturbation. In order to model an N-axis robot manipulator with n revolute joints assume that: $\hat{q}_i \;:\; i = 1, 2, \ldots, n$ denote the position of ith link and $\hat{q}_i \;:\; i = n+1, n+2, \ldots, 2n$ denote the position of the ith actuator scaled by the actuator gear ratio. If the joint is rigid $\hat{q}_i = \hat{q}_{n+i} \forall i$. For a flexible joint, if the flexibility is modeled with a linear torsional spring with constant k_i, the elastic force z_i is derived from:

$$z_i = k_i(\hat{q}_i - \hat{q}_{n+i}). \tag{1}$$

The spring constants k_is are relatively large and rigidity is modeled by the limit $k_i \to \infty$. Let u_i denotes the generalized force applied by the ith actuator and use the notation:

$$q = (\hat{q}_1, \ldots, \hat{q}_n, \hat{q}_{n+1}, \ldots, \hat{q}_{2n})^{\mathrm{T}} = \left(q_1^{\mathrm{T}} | q_2^{\mathrm{T}}\right)^{\mathrm{T}}. \tag{2}$$

The equation of motion of the system can be written in the following form using the Euler–Lagrange formulation:

$$\begin{cases} M(q_1)\ddot{q}_1 + N(q_1, \dot{q}_1) = K(q_2 - q_1) \\ J\ddot{q}_2 = K(q_1 - q_2) - D\dot{q}_2 + T_F + u \end{cases} \tag{3}$$

in which:

$$N(q_1, \dot{q}_1) = V_{\mathrm{m}}(q_1, \dot{q}_1)\dot{q}_1 + G(q_1) + F_{\mathrm{d}}\dot{q}_1 + F_{\mathrm{s}}(\dot{q}_1) + T_{\mathrm{d}}, \tag{4}$$

and K is the joint stiffness matrix, $M(q_1)$ is the mass matrix, $V_{\mathrm{m}}(q_1, \dot{q}_1)$ is the matrix of coriolis and centrifugal terms, $G(q_1)$ is the vector of gravity terms, F_{d} is the viscous friction matrix, $F_{\mathrm{s}}(\dot{q}_1)$ is the coulomb friction vector, T_{d} is the vector of the joint bounded unmodeled dynamics, J is the actuator moments of the inertia matrix, D is the actuator viscous friction matrix and T_{F} is the actuator bounded unmodeled dynamics. For all revolute manipulators, it is shown in Refs [15, 17] that:

$$m_1 I \leqslant M(q_1) \leqslant m_2 I, \quad \|V_{\mathrm{m}}(q_1, \dot{q}_1)\| \leqslant \zeta_{\mathrm{c}} \|\dot{q}_1\|, \tag{5}$$

$$\|G(q_1)\| \leqslant \zeta_{\mathrm{g}}, \quad \|F_{\mathrm{d}}\dot{q}_1 + F_{\mathrm{s}}(\dot{q}_1)\| = \zeta_{\mathrm{f0}} + \zeta_{\mathrm{f1}} \|q_1\|, \tag{6}$$

$$j_1 I \leqslant J \leqslant j_2 I, \quad d_1 I \leqslant D \leqslant d_2 I. \tag{7}$$

Moreover, if the perturbations are bounded then:

$$\|T_{\rm d}\| \leqslant \zeta_{\rm e}, \quad \|T_{\rm F}\| \leqslant \zeta_{\rm f2} \tag{8}$$

in which $\zeta_{\rm f2}, \zeta_{\rm e}, d_2, d_1, j_2, j_1, \zeta_{\rm f1}, \zeta_{\rm f0}, \zeta_{\rm g}, \zeta_{\rm c}, m_2$ and m_1 are positive real constants. If the joints are all rigid:

$$M_{\rm t}(q)\ddot{q} + N_{\rm t}(q, \dot{q}) = u_0 \tag{9}$$

in which $q = q_1$ and $M_{\rm t}$ is a positive definite matrix. This model is the model of a FJR where $k \to \infty$ verifying that the FJR model is a singularly perturbed model of rigid system. Assume that all spring constants are equal (this assumption does not reduce the generality of the formulation, since by scaling z we reach the same conclusion), the elastic forces of the springs can be calculated by:

$$z = k(q_1 - q_2), \quad K = kI \tag{10}$$

in order to use a small quantity for a singular perturbation define $\epsilon = 1/k$ by which for a rigid system ($k \to \infty$) in this form we have $\epsilon \to 0$. Multiplying M^{-1} to both sides of (3) and taking $z = k(q_1 - q_2)$, $q = q_1$ and using $\dot{q}_2 = \dot{q}_1 - \epsilon\dot{z}$:

$$\begin{cases} \ddot{q} = a_1(q, \dot{q}) + A_1(q)z \\ \epsilon\ddot{z} = a_2(q, \dot{q}, \epsilon\dot{z}) + A_2(q)z + B_2 u \end{cases} \tag{11}$$

in which:

$$A_1 = -M^{-1}(q), \quad a_1 = -M^{-1}(q)N(q, \dot{q}), \tag{12}$$

$$a_2 = -\epsilon J^{-1}D\dot{z} + J^{-1}D\dot{q} - J^{-1}T_{\rm F} - M^{-1}(q)N(q, \dot{q}), \tag{13}$$

$$A_2 = -\big(M^{-1}(q) + J^{-1}\big), \quad B_2 = -J^{-1}. \tag{14}$$

Equation (11) represents a FJR as a non-linear and coupled system. This representation includes both rigid and flexible subsystems in the form of a singular perturbation model.

3. REDUCED FLEXIBLE MODEL

The singular perturbation model of the FJR is given in (11). This model represents the flexibility in the joints; however, the reduced-order model is the model of a rigid system, which can be easily derived from (11) by setting $\epsilon = 0$. With some matrix manipulation it can be shown that:

$$(M + J)\ddot{q} + N - T_{\rm F} + D\dot{q} = u_0.$$

Rewrite this equation in this form:

$$M_{\rm t}(q)\ddot{q} + N_{\rm t}(q, \dot{q}) = u_0 \tag{15}$$

in which:

$$M_t(q) = M(q) + J, \tag{16}$$

$$\begin{aligned} N_t(q,\dot{q}) &= N(q,\dot{q}) - T_F + D\dot{q} \\ &= V_m(q,\dot{q})\dot{q} + G(q) + (F_d + D)\dot{q} + F_s(\dot{q}) + T_d - T_F. \end{aligned} \tag{17}$$

This representation introduces a $2n$ dimension manifold, M_o, which is called the rigid manifold. If $\epsilon \neq 0$ the produced manifold M_ϵ, which is a function of ϵ, represents the flexible system. To define the flexible manifold M_ϵ assume:

$$z = H(q,\dot{q},u,\epsilon) \quad q \in R^n, u \in R^n, z \in R^n, \tag{18}$$

$$\dot{z} = \dot{H}(q,\dot{q},u,\epsilon) \quad q \in R^n, u \in R^n, z \in R^n, \tag{19}$$

M_ϵ is an integral manifold for the flexible system if for each initial condition:

$$\begin{cases} z(t) = \Delta \\ \dot{z}(t) = \Delta' \end{cases} \quad \text{and} \quad \begin{cases} q(t) = \zeta \\ \dot{q}(t) = \zeta' \end{cases}$$

in M_ϵ all trajectories of $q(t)$ and $z(t)$ for $t > t_o$ remain in the manifold M_ϵ. In other words $\forall t > t_o$:

$$z(t) = H(q(t),\dot{q}(t),u(t),\epsilon), \tag{20}$$

$$\dot{z}(t) = \dot{H}(q(t),\dot{q}(t),u(t),\epsilon). \tag{21}$$

Equations (20) and (21) are called the manifold conditions. An integral manifold for a FJR exists if $A_2 = -(M^{-1} + J^{-1})$ is non-singular $\forall q \in R^n$ [2]. This is always true since the mass matrices M and J are positive definite. If the manifold conditions are not satisfied at the initial time t_o, but the fast dynamics are asymptotically stable, the initial transient will die down shortly and the manifold condition will be satisfied after a short transient.

In order to derive the reduced flexible model, the flexible manifold is used in the formulation. Assume that the function H is several times differentiable with respect to its arguments. Hence, by differentiating (20) and (21), and substitution in (11):

$$\epsilon\ddot{H}(q,\dot{q},u,\epsilon) = a_2\big(q,\dot{q},\epsilon\dot{H}(q,\dot{q},u,\epsilon)\big) + A_2(q)H(q,\dot{q},u,\epsilon) + B_2u \tag{22}$$

in which:

$$\dot{H} = \left(\frac{\partial H}{\partial q} + \frac{\partial H}{\partial u}\frac{\partial u}{\partial q}\right)\dot{q} + \frac{\partial H}{\partial \dot{q}}(a_1 + A_1H) + \frac{\partial H}{\partial u}\frac{\partial u}{\partial t}. \tag{23}$$

Now, the reduced flexible model can be derived by replacing z, $\dot{z}$ with H, $\dot{H}$ in (11):

$$\ddot{q} = a_1(q,\dot{q}) + A_1(q)H(q,\dot{q},u,\epsilon). \tag{24}$$

The order of this equation is equal to the rigid system; however, this model includes the effects of flexibility in the form of an invariant integral manifold embedded in itself. Hence, this reduced order model is not an approximation of the FJR model, but it represents its projection on the integral manifold.

4. COMPOSITE CONTROL

In order that the reduced flexible model holds for the system, it is essential that M_ϵ be an invariant manifold or the fast dynamics be asymptotically stable. This can be satisfied using a composite control scheme [2]. In this framework the control effort u consists of two main parts, i.e. u_s the control effort for slow subsystem and u_f the control effort for fast subsystem, as:

$$u = u_s(q, \dot{q}, \epsilon) + u_f(\eta, \dot{\eta}) \tag{25}$$

in which $u_f(\eta, \dot{\eta})$ is designed such that the fast dynamics becomes asymptotically stable. η denotes the deviations of fast state variables from the integral manifold:

$$\eta = z - H(q, \dot{q}, u_s, \epsilon), \tag{26}$$

$$\dot{\eta} = \dot{z} - \dot{H}(q, \dot{q}, u_s, \epsilon). \tag{27}$$

The slow component of the control effort, $u_s(q, \dot{q}, \epsilon)$, is also designed based on the reduced flexible model. We describe the design technique for u_f and u_s in the next subsections.

4.1. Fast subsystem dynamics and control

Recall (26) and differentiate twice:

$$\begin{aligned}\epsilon\ddot{\eta} &= \epsilon\ddot{z} - \epsilon\ddot{H}\\ &= a_2(q, \dot{q}, \epsilon\dot{z}) + A_2(q)z + B_2u - \big(a_2\big(q, \dot{q}, \epsilon\dot{H}\big) + A_2(q)H + B_2u_s\big)\end{aligned}$$

or:

$$\epsilon\ddot{\eta} = \big[a_2(q, \dot{q}, \epsilon\dot{z}) - a_2\big(q, \dot{q}, \epsilon\dot{H}\big)\big] + A_2(q)\eta + B_2u_f. \tag{28}$$

Substituting the value of a_2 and using fast time scale $\tau = t/\sqrt{\epsilon}$ with some manipulations this leads to [18]:

$$\epsilon\ddot{\eta} = A_2(q)\eta + B_2u_f, \tag{29}$$

and in state space form:

$$\epsilon\begin{bmatrix}\dot{\eta}\\ \ddot{\eta}\end{bmatrix} = \begin{bmatrix}\emptyset & \epsilon I\\ A_2(q) & \emptyset\end{bmatrix}\begin{bmatrix}\eta\\ \dot{\eta}\end{bmatrix} + \begin{bmatrix}\emptyset\\ B_2\end{bmatrix}u_f. \tag{30}$$

The flexible modes are not stable since the eigenvalues are on the imaginary axis. Hence, u_f must be designed such that the eigenvalues are shifted to the open left half plane in order to guarantee stability.

THEOREM 1. *The diagonal and positive definite matrices K_{pf} and K_{vf} exist such that the closed loop system including the subsystem* (29) *with the control effort $u_f = K_{pf}\eta + K_{vf}\dot{\eta}$ becomes globally asymptotically stable.* [*Proof in Ref.* [19].]

4.2. Control of the reduced flexible model

The reduced flexible model represents the effect of flexibility in the form of the flexible integral manifold. In this section a robust control algorithm is proposed for the system based on this model. In order to accurately derive a robust control law $u_s(q, \dot{q}, \epsilon)$ for the system, manipulation of the partial differential equation is necessary. To avoid complex manipulations, we propose deriving the robust control law $u_s(q, \dot{q}, \epsilon)$ to any order of ϵ from the series expansion of the integral manifold to the same order of ϵ:

$$H(q, \dot{q}, u_s, \epsilon) = H_0(q, \dot{q}, u_s) + \epsilon H_1(q, \dot{q}, u_s) + \cdots \tag{31}$$

and implement the controller $u_s(q, \dot{q}, \epsilon)$ in the same form as:

$$u_s(q, \dot{q}, \epsilon) = u_0(q, \dot{q}) + \epsilon u_1(q, \dot{q}) + \cdots \tag{32}$$

in which the functions $H_i(q, \dot{q}, u_s)$, $u_i(q, \dot{q})$, $i = 0, 1, \ldots$ are calculated iteratively without the need to solve the partial differential equations. It is important to note that as $\epsilon \to 0$, u_s tends to rigid control and H tends to rigid integral manifold. By substitution of (31) and (32) into manifold condition (22) we reach:

$$\begin{aligned} &\epsilon \ddot{H}_0(q, \dot{q}, u_s) + \epsilon^2 \ddot{H}_1(q, \dot{q}, u_s) + \cdots = \\ &\quad a_2\big(q, \dot{q}, \epsilon \dot{H}_0 + \epsilon^2 \dot{H}_1 + \cdots\big) + A_2(q)(H_0 + \epsilon H_1 + \cdots) \\ &\quad + B_2(u_0 + \epsilon u_1 + \cdots). \end{aligned} \tag{33}$$

The right-hand side of (33) can be expanded with respect to the powers of ϵ; and by addition of equal powers of ϵ; a set of equations for H_i, u_i, $i = 0, 1, \ldots$ in terms of ϵ are the result. The first-order approximation of (33) will result in:

$$\begin{aligned} \epsilon \ddot{H}_0(q, \dot{q}, u_s) = a_2\big(q, \dot{q}, \epsilon \dot{H}_0\big) + A_2(q)(H_0 + \epsilon H_1) \\ + B_2(u_0 + \epsilon u_1) + O\big(\epsilon^2\big). \end{aligned} \tag{34}$$

When $\epsilon = 0$ the equation relating u_0 to H_0 will be:

$$0 = a_{20} + A_2(q) H_0(q, \dot{q}, u_0) + B_2 u_0 \tag{35}$$

in which:

$$a_{20} = a_2(q, \dot{q}, 0) = J^{-1} D\dot{q} - J^{-1} T_F(q, \dot{q}) - M^{-1}(q) N(q, \dot{q}), \tag{36}$$

u_0 is designed using a robust design technique based on the rigid reduced order model ($\epsilon = 0$) and H_0 is calculated from:

$$H_0 = -A_2^{-1}(a_{20} + B_2 u_0). \tag{37}$$

The details of the robust design technique are explained in the next section. Now, since u_0 and H_0 are known from (34), H_1 can be similarly calculated in terms of u_1 and the first-order manifold H_1 can be substituted into the reduced flexible model (24). If higher-order terms are neglected, the first-order corrected model for the system is derived.

In order to calculate H_1 from H_0 and u_0, let:

$$a_2\big(q, \dot{q}, \epsilon \dot{H}\big) = a_{20} + \epsilon \Delta a_2 + O\big(\epsilon^2\big)$$

in which a_{20} is given in (36) and comparing to (13) we reach:

$$\begin{cases} \Delta a_2 = -J^{-1} D\dot{H} \\ \Delta a_{20} = -J^{-1} D\dot{H}_0. \end{cases}$$

Hence:

$$\epsilon \ddot{H}_0 = a_{20} + A_2 H_0 + B_2 u_0 + \epsilon(\Delta a_{20} + A_2 H_1 + B_2 u_1) + O\big(\epsilon^2\big). \tag{38}$$

Comparing (38) to (35):

$$\ddot{H}_0 = \Delta a_{20} + A_2 H_1 + B_2 u_1. \tag{39}$$

Therefore:

$$H_1 = A_2^{-1}\big(\ddot{H}_0 - \Delta a_{20} - B_2 u_1\big). \tag{40}$$

To calculate u_1 refer to reduced flexible model (24) and approximate it to the first power of ϵ:

$$\ddot{q} = a_1(q, \dot{q}) + A_1(q) H_0 + \epsilon A_1(q) A_2^{-1}\big(\ddot{H}_0 - \Delta a_{20} - B_2 u_1\big).$$

By factoring the equal powers of ϵ we reach:

$$u_1 = B_2^{-1}\big(\ddot{H}_0 - \Delta a_{20}\big). \tag{41}$$

The only condition on robust control design is that u_0 must be at least twice differentiable. Finally, the control law for the slow subsystem has the form:

$$u_s = u_0 + \epsilon u_1. \tag{42}$$

In which u_1 is called the corrective term that is derived through this subsection and u_0 is the robust control based on the rigid model elaborated in the next section.

4.3. Robust PID control for the rigid model

In this section we first propose a robust PID controller based on the rigid model of the system and then prove its robust stability with respect to the model uncertainties. Recall the rigid model of the system from (15) and choose a PID controller for u_0:

$$u_0 = K_V \dot{e} + K_P e + K_I \int_0^t e(s)\,\mathrm{d}s = Kx \tag{43}$$

in which

$$\begin{cases} e = q_d - q \\ K = [K_I \; K_P \; K_V] \\ x = \left[\int_0^t e^{\mathrm{T}}(s)\,\mathrm{d}s \; e^{\mathrm{T}} \; \dot{e}^{\mathrm{T}}\right]^{\mathrm{T}}. \end{cases}$$

Similar to Refs [16, 17], assume:

$$\underline{m}_{\mathrm{t}} I \leqslant M_{\mathrm{t}}(q) \leqslant \overline{m}_{\mathrm{t}} I, \tag{44}$$

and put some limits on:

$$\|N_{\mathrm{t}}\| \leqslant \beta_0 + \beta_1\|L\| + \beta_2\|L\|^2, \quad \|V_{\mathrm{m}}\| \leqslant \beta_3 + \beta_4\|L\| \tag{45}$$

in which $\|\cdot\|$ is the Euclidean norm and $L = [e^{\mathrm{T}}\ \dot{e}^{\mathrm{T}}]$. Implement the control law u_0 in (15) to get:

$$\dot{x} = Ax + B\Delta A, \tag{46}$$

where:

$$A = \begin{bmatrix} \emptyset & I_n & \emptyset \\ \emptyset & \emptyset & I_n \\ -M_{\mathrm{t}}^{-1}K_I & -M_{\mathrm{t}}^{-1}K_P & -M_{\mathrm{t}}^{-1}K_V \end{bmatrix} B = \begin{bmatrix} \emptyset \\ \emptyset \\ M_{\mathrm{t}}^{-1} \end{bmatrix}$$

$$\Delta A = N_{\mathrm{t}} + M_{\mathrm{t}}\ddot{q}_{\mathrm{d}}. \tag{47}$$

To analyze the system's robust stability, consider the following Lyapunov function:

$$V(x) = x^{\mathrm{T}}Px = \frac{1}{2}\left[\alpha_2\int_0^t e(s)\,\mathrm{d}s + \alpha_1 e + \dot{e}\right]^{\mathrm{T}} \cdot M_{\mathrm{t}}$$
$$\left[\alpha_2\int_0^t e(s)\,\mathrm{d}s + \alpha_1 e + \dot{e}\right] + w^{\mathrm{T}}P_1 w \tag{48}$$

in which:

$$w = \begin{bmatrix} \int_0^t e(s)\,\mathrm{d}s \\ e \end{bmatrix} P_1 = \frac{1}{2}\begin{bmatrix} \alpha_2 K_P + \alpha_1 K_I & \alpha_2 K_V + K_I \\ \alpha_2 K_V + K_I & \alpha_1 K_V + K_P \end{bmatrix}.$$

Hence:

$$P = \frac{1}{2}\begin{bmatrix} \alpha_2 K_P + \alpha_1 K_I + \alpha_2^2 M_{\mathrm{t}} & \alpha_2 K_V + K_I + \alpha_1\alpha_2 M_{\mathrm{t}} & \alpha_2 M_{\mathrm{t}} \\ \alpha_2 K_V + K_I + \alpha_1\alpha_2 M_{\mathrm{t}} & \alpha_1 K_V + K_P + \alpha_1^2 M_{\mathrm{t}} & \alpha_1 M_{\mathrm{t}} \\ \alpha_2 M_{\mathrm{t}} & \alpha_1 M_{\mathrm{t}} & M_{\mathrm{t}} \end{bmatrix}.$$

Since M_{t} is a positive definite matrix, P is positive definite, if and only if, P_1 is positive definite. Now choose:

$$\begin{cases} K_P = k_P I \\ K_V = k_V I \\ K_I = k_I I \end{cases}$$

such that:

$$\begin{bmatrix} \alpha_2 k_P + \alpha_1 k_I & \alpha_2 k_V + k_I \\ \alpha_2 k_V + k_I & \alpha_1 k_V + k_P \end{bmatrix}$$

becomes positive definite. The following Lemma gives the conditions where V can become positive definite and upper and lower bounded.

LEMMA 1. *Assume the following inequalities hold*:

$$\alpha_1 > 0 \quad \alpha_2 > 0 \quad \alpha_1 + \alpha_2 < 1$$
$$s_1 = \alpha_2(k_P - k_V) - (1 - \alpha_1)k_I - \alpha_2(1 + \alpha_1 - \alpha_2)\overline{m}_t > 0$$
$$s_2 = k_P + (\alpha_1 - \alpha_2)k_V - k_I - \alpha_1(1 + \alpha_2 - \alpha_1)\overline{m}_t > 0$$

Then P is positive definite and satisfies the following inequality (Rayleigh–Ritz) [15]:

$$\underline{\lambda}(P)\|x\|^2 \leqslant V(x) \leqslant \overline{\lambda}(P)\|x\|^2 \tag{49}$$

in which:

$$\underline{\lambda}(P) = \min\left\{\frac{1 - \alpha_1 - \alpha_2}{2}\underline{m}_t, \frac{s_1}{2}, \frac{s_2}{2}\right\}$$
$$\overline{\lambda}(P) = \max\left\{\frac{1 + \alpha_1 + \alpha_2}{2}\overline{m}_t, \frac{s_3}{2}, \frac{s_4}{2}\right\}$$

and:

$$s_3 = \alpha_2(k_P + k_v) + (1 + \alpha_1)k_I + (1 + \alpha_1 + \alpha_2)\alpha_2\overline{m}_t$$
$$s_4 = \alpha_1\overline{m}_t(1 + \alpha_1 + \alpha_2) + (\alpha_1 + \alpha_2)k_V + k_P + k_I.$$

Proof is based on Gershgorin theorem and is similar to that in [16].

Now when P is positive definite then:

$$\begin{aligned}
\dot{V}(x) &= x^T(A^T P + PA + \dot{P})x + 2x^T PB\Delta A \\
&= -x^T Qx + \frac{1}{2}x^T \begin{bmatrix} \alpha_2 I \\ \alpha_1 I \\ I \end{bmatrix} \dot{M}_t [\,\alpha_2 I \quad \alpha_1 I \quad I\,] x \\
&\quad + x^T \begin{bmatrix} \alpha_2 I \\ \alpha_1 I \\ I \end{bmatrix} \Delta A + \frac{1}{2}x^T \begin{bmatrix} \emptyset & \alpha_2^2 I & \alpha_1\alpha_2 I \\ \alpha_2 I & 2\alpha_1\alpha_2 I & (\alpha_1^2 + \alpha_2)I \\ \alpha_1\alpha_2 I & (\alpha_1^2 + \alpha_2)I & \alpha_1 I \end{bmatrix} \\
&\quad \times \begin{bmatrix} M_t & \emptyset & \emptyset \\ \emptyset & M_t & \emptyset \\ \emptyset & \emptyset & M_t \end{bmatrix} x
\end{aligned} \tag{50}$$

refer to [15]:

$$y^T \dot{M}_t y = 2y^T V_m y$$

with some manipulations we can show [20]:

$$\dot{V}(x) \leqslant -\gamma\|x\|^2 + \lambda_1\|V_m\|\|x\|^2 + \lambda_2\overline{m}_t\|x\|^2 + \alpha_2^{-1}\lambda_1\|x\|\|\Delta A\|$$

$$\dot{V}(x) \leqslant \|x\|(\xi_0 - \xi_1\|x\| + \xi_2\|x\|^2), \tag{51}$$

and:

$$\gamma = \min\{\alpha_2 k_I, \alpha_1 k_P - \alpha_2 k_V - k_I, k_V\}.$$

Now considering (45), (47) and (51), and $\|L\| \leqslant \|x\|$ then:

$$\xi_0 = \alpha_2^{-1}\lambda_1\beta_0 + \alpha_2^{-1}\lambda_1\lambda_3\overline{m}_{\mathrm{t}}$$
$$\xi_1 = \gamma - \lambda_1\beta_3 - \lambda_2\overline{m}_{\mathrm{t}} - \alpha_2^{-1}\lambda_1\beta_1$$
$$\xi_2 = \lambda_1\beta_4 + \alpha_2^{-1}\lambda_1\beta_2$$

in which:

$$\begin{cases} \lambda_1 = \lambda_{\max}(R_1) \\ \lambda_2 = \lambda_{\max}(R_2) \\ \lambda_3 = \sup\|\ddot{q}_{\mathrm{d}}\| \end{cases}$$

and $\lambda_{\min}$ and $\lambda_{\max}$ are the least and largest eigenvalues, respectively, and:

$$R_1 = \begin{bmatrix} \alpha_2^2 I & \alpha_1\alpha_2 I & \alpha_2 I \\ \alpha_1\alpha_2 I & \alpha_1^2 I & \alpha_1 I \\ \alpha_2 I & \alpha_1 I & I \end{bmatrix}$$
$$R_2 = \frac{1}{2}\begin{bmatrix} \emptyset & \alpha_2^2 I & \alpha_1\alpha_2 I \\ \alpha_2 I & 2\alpha_1\alpha_2 I & (\alpha_1^2 + \alpha_2)I \\ \alpha_1\alpha_2 I & (\alpha_1^2 + \alpha_2)I & \alpha_1 I \end{bmatrix}.$$

According to the result obtained so far, we can prove the stability of the error system based on the following theorem.

THEOREM 2. *The error system* (46) *is UUB stable if* ξ_1 *is chosen large enough.*

Proof. According to (51) and (49), and Lemma 3.5 from Ref. [15], if the following condition holds, the system is UUB stable with respect to $B(0, d)$, where:

$$d = \frac{2\xi_0}{\xi_1 + \sqrt{\xi_1^2 - 4\xi_0\xi_2}}\sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)}}.$$

The conditions are:

$$\xi_1 > 2\sqrt{\xi_0\xi_2}$$
$$\xi_1^2 + \xi_1\sqrt{\xi_1^2 - 4\xi_0\xi_2} > 2\xi_0\xi_2\left(1 + \sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)}}\right)$$
$$\xi_1 + \sqrt{\xi_1^2 - 4\xi_0\xi_2} > 2\xi_2\|x_0\|\sqrt{\frac{\overline{\lambda}(P)}{\underline{\lambda}(P)}}.$$

These conditions can be simply met by making ξ_1 large enough by choosing large enough control gains K_P, K_V and K_I. □

5. STABILITY ANALYSIS OF THE COMPLETE CLOSED-LOOP SYSTEM

The stability of the fast and slow subsystems were analyzed separately in previous sections. However, the stability of the complete closed-loop system may not be guaranteed through these separate analysis [2]. In this section the stability of the complete system is analyzed. Recall the dynamic equations of the FJR (11). The integral manifold and the control effort are chosen as:

$$\eta = z - H$$
$$H = H_0 + \epsilon H_1$$
$$u = u_s + u_f = u_0 + \epsilon . u_1 + u_f .$$

Combine these equations to (11), (40), (37) and (43), and consider $x = [\,\int_0^t e(s)^T \mathrm{d}s \quad e^T \quad \dot{e}^T\,]^T$; $y = [\,\eta^T \quad \dot{\eta}^T\,]^T$, then:

$$\dot{x} = Ax + B\Delta A + C\,[\,I \quad \emptyset\,]\,y, \tag{52}$$
$$\epsilon \dot{y} = \tilde{A} y \tag{53}$$

in which:

$$A = \begin{bmatrix} \emptyset & I & \emptyset \\ \emptyset & \emptyset & I \\ -M_t^{-1} K_I & -M_t^{-1} K_P & -M_t^{-1} K_V \end{bmatrix}, \quad B = \begin{bmatrix} \emptyset \\ \emptyset \\ M_t^{-1} \end{bmatrix},$$

$$\Delta A = N_t + M_t \ddot{q}_d ,$$

$$C = \begin{bmatrix} \emptyset \\ \emptyset \\ -A_1 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} \emptyset & \epsilon I \\ A_2 + B_2 K_{pf} & -\epsilon J^{-1} D + B_2 K_{vf} \end{bmatrix}.$$

THEOREM 3. *There exist diagonal and positive definite matrices K_{pf} and K_{vf} such that the closed-loop system* (53) *becomes globally asymptotically stable.*

Proof. Substitute A_2 and B_2 from (14) into (53), and define:

$$M^{-1} + J^{-1} + J^{-1} K_{pf} = U$$
$$\epsilon J^{-1} D + J^{-1} K_{vf} = G.$$

Where U and G are both positive definite, since M, J, K_{pf} and K_{vf} are all positive definite, hence:

$$\epsilon \begin{bmatrix} \dot{\eta} \\ \ddot{\eta} \end{bmatrix} = \begin{bmatrix} \emptyset & \epsilon I \\ -U & -G \end{bmatrix} \begin{bmatrix} \eta \\ \dot{\eta} \end{bmatrix}.$$

Consider the following Lyapunov function:

$$V_F = y^T S y$$

in which $y = [\eta \quad \dot{\eta}]^T$ and:

$$S = \frac{1}{2} \begin{bmatrix} \frac{2}{\epsilon} I & G^{-1} \\ G^{-1} & U^{-1} \end{bmatrix}.$$

In order to have positive definite S, according to the Shur complement we must have:

$$\begin{cases} \dfrac{2}{\epsilon} I > 0 \\ U^{-1} - G^{-1}\left(\dfrac{2}{\epsilon} I\right)^{-1} G^{-1} > 0 \Longrightarrow U^{-1} - \dfrac{\epsilon}{2} G^{-2} > 0. \end{cases} \tag{54}$$

Now since U^{-1} and G^{-2} are positive definite in order to satisfy (54) the following condition must be met:

$$\epsilon < \frac{2\lambda_{\min}(U^{-1})}{\lambda_{\max}(G^{-2})}$$

in which $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and the largest Eigenvalue, respectively. Differentiate V_F along trajectories of (53):

$$\begin{aligned} \dot{V}_F &= \dot{y}^{\mathrm{T}} S y + y^{\mathrm{T}} S \dot{y} + y^{\mathrm{T}} \dot{S} y \\ &= -\frac{1}{\epsilon} \eta^{\mathrm{T}} G^{-1} U \eta - \dot{\eta}^{\mathrm{T}} \left[\frac{1}{\epsilon} U^{-1} G - \frac{1}{2} (G^{-1} + (U^{-1})^{\mathrm{T}}) \right] \dot{\eta} < 0. \end{aligned}$$

Since G is diagonal and positive definite, and limited $q, \dot{q}$ will limit $(U^{-1})'$, then by choosing appropriate values for K_{vf} and K_{pf}, $\dot{V}_F$ becomes negative definite and it can be written as:

$$\dot{V}_F = -y^{\mathrm{T}} W y$$

in which:

$$W = \frac{1}{2} \begin{bmatrix} \dfrac{1}{\epsilon}(G^{-1} U + U G^{-1}) & \emptyset \\ \emptyset & \dfrac{1}{\epsilon}(U^{-1} G + G U^{-1}) - (U^{-1})^{\mathrm{T}} - G^{-1} \end{bmatrix}.$$

□

THEOREM 4. *The closed-loop system of* (52) *and* (53) *is UUB stable if* K_{pf}, K_{vf} *and* ξ_1 *are chosen large enough.*

Proof. Consider the following composite Lyapunov function:

$$V(x, y) = x^{\mathrm{T}} P x + y^{\mathrm{T}} S y, \tag{55}$$

where $x^{\mathrm{T}} P x$ is the Lyapunov function candidated for slow subsystem, and $y^{\mathrm{T}} S y$ is the Lyapunov function of fast subsystem (53). Therefore, from the Rayleigh–Ritz inequality:

$$\underline{\lambda}(P) \|x\|^2 \leqslant x^{\mathrm{T}} P x \leqslant \overline{\lambda}(P) \|x\|^2$$
$$\underline{\lambda}(S) \|y\|^2 \leqslant y^{\mathrm{T}} S y \leqslant \overline{\lambda}(S) \|y\|^2$$

in which $\overline{\lambda}$ and $\underline{\lambda}$ are the largest and smallest eigenvalues, respectively. Adding the above inequalities:

$$\underline{\lambda}(S)\|y\|^2 + \underline{\lambda}(P)\|x\|^2 \leqslant V(x, y) \leqslant \overline{\lambda}(S)\|y\|^2 + \overline{\lambda}(P)\|x\|^2.$$

Define:

$$Z_t = [\ \|x\| \quad \|y\|\ ]^{\mathrm{T}} \tag{56}$$

then:

$$\begin{aligned}[\ \|x\| \quad \|y\|\ ]\begin{bmatrix} \underline{\lambda}(P) & 0 \\ 0 & \underline{\lambda}(S) \end{bmatrix}\begin{bmatrix} \|x\| \\ \|y\| \end{bmatrix} &\leqslant V(x, y) \\ \leqslant [\ \|x\| \quad \|y\|\ ]\begin{bmatrix} \overline{\lambda}(P) & 0 \\ 0 & \overline{\lambda}(S) \end{bmatrix}\begin{bmatrix} \|x\| \\ \|y\| \end{bmatrix}&.\end{aligned}$$

Again apply the Rayleigh–Ritz inequality:

$$\underline{\lambda}\|Z_{\mathrm{t}}\| \leqslant V(Z_{\mathrm{t}}) \leqslant \overline{\lambda}\|Z_{\mathrm{t}}\|, \tag{57}$$

where:

$$\begin{aligned}\underline{\lambda} &= \min\{\underline{\lambda}(P), \underline{\lambda}(S)\}, \\ \overline{\lambda} &= \max\{\overline{\lambda}(P), \overline{\lambda}(S)\}.\end{aligned}$$

Now differentiate (55) along the trajectories of (52) and (53):

$$\begin{aligned}\dot{V} = 2x^{\mathrm{T}}P\dot{x} + x^{\mathrm{T}}\dot{P}x + 2y^{\mathrm{T}}S\dot{y} + y^{\mathrm{T}}\dot{S}y &= 2x^{\mathrm{T}}PC[I \quad \emptyset]y \\ + [2x^{\mathrm{T}}P(Ax + B\Delta A) + x^{\mathrm{T}}\dot{P}x] &+ 2y^{\mathrm{T}}S\dot{y} + y^{\mathrm{T}}\dot{S}y,\end{aligned}$$

and consider (50) and (51), and define $\gamma_1 = \lambda_{\max}(M^{-1})$. As shown in Theorem 3:

$$2y^{\mathrm{T}}S\dot{y} + y^{\mathrm{T}}\dot{S}y \leqslant -\lambda_{\min}(W)\|y\|^2.$$

Hence:

$$\begin{aligned}\dot{V} \leqslant -[\ \|x\| \quad \|y\|\ ]&\begin{bmatrix} \xi_1 & -\gamma_1\overline{\lambda}(P) \\ -\gamma_1\overline{\lambda}(P) & \lambda_{\min}(W) \end{bmatrix}\begin{bmatrix} \|x\| \\ \|y\| \end{bmatrix} \\ + \xi_0\|x\| + \xi_2\|x\|^3&,\end{aligned}$$

and according to (56):

$$\dot{V} \leqslant -Z_{\mathrm{t}}^{\mathrm{T}}RZ_{\mathrm{t}} + \xi_0\|Z_{\mathrm{t}}\| + \xi_2\|Z_{\mathrm{t}}\|^3,$$

where:

$$R = \begin{bmatrix} \xi_1 & -\gamma_1\overline{\lambda}(P) \\ -\gamma_1\overline{\lambda}(P) & \lambda_{\min}(W) \end{bmatrix}.$$

In order to have positive definite R:

$$\Delta_1 : \xi_1\lambda_{\min}(W) - \gamma_1^2\overline{\lambda}^2(P) > 0$$

or:

$$\lambda_{\min}(W) > \frac{\gamma_1^2\overline{\lambda}^2(P)}{\xi_1}. \tag{58}$$

Condition (58) is met by choosing appropriate $K_{\rm pf}$ and $K_{\rm vf}$ for fast subsystem, hence:

$$\dot{V} \leqslant \|Z_{\rm t}\|(\xi_0 - \lambda_{\min}(R)\|Z_{\rm t} + \xi_2\|Z_{\rm t}\|^2). \tag{59}$$

Now, according to (59) and (57), and Lemma 3.5 of [15], if these conditions are met then the closed-loop system is UUB stable with respect to $Y(0, d')$, where:

$$d' = \frac{2\xi_0}{\lambda_{\min}(R) + \sqrt{\lambda_{\min}^2(R) - 4\xi_0\xi_2}}\sqrt{\frac{\overline{\lambda}}{\underline{\lambda}}},$$

and the stability conditions are:

$$\lambda_{\min}(R) > 2\sqrt{\xi_0\xi_2}$$

$$\lambda_{\min}^2(R) + \lambda_{\min}(R)\sqrt{\lambda_{\min}^2(R) - 4\xi_0\xi_2} > 2\xi_0\xi_2\left(1 + \sqrt{\frac{\overline{\lambda}}{\underline{\lambda}}}\right)$$

$$\lambda_{\min}(R) + \sqrt{\lambda_{\min}^2(R) - 4\xi_0\xi_2} > 2\xi_2\|Z_{\rm t0}\|\sqrt{\frac{\overline{\lambda}}{\underline{\lambda}}}.$$

□

These conditions are simply met by increasing $\lambda_{\min}(R)$, through appropriate choice of large ξ_1, and $\lambda_{\min}(W)$. ξ_1 is a function of the robust PID gains K_P, K_I and K_V, and $\lambda_{\min}(W)$ are affected by the fast subsystem gains $K_{\rm pf}$ and $K_{\rm vf}$. Therefore, the robust stability of the closed-loop system is guaranteed by the choice of the controller gains such that the above conditions are met.

6. SIMULATIONS

A simulation study has been performed in order to verify the effectiveness of the algorithm. In the following simulation study, the results of the closed-loop performance of two flexible joint manipulators examined in the literature is compared to that of the proposed control algorithm. First, a single-joint manipulator examined in detail by Spong [4] has been simulated and the closed-loop performances are compared. Then, the two-link manipulator studied by Al-Ashoor *et al.* [6] is examined in detail and robust PID controller is designed for each joint. Moreover, the closed-loop performance of this system is presented. The simulation results show the effectiveness of the proposed algorithm, despite the simplicity of its structure and the convenience of its online implementation.

6.1. Single-link flexible joint manipulator

Consider the single-link flexible joint manipulator introduced in Ref. [4], whose parameters are given in Table 1. The dynamic equation of motion of this system is as follows:

Table 1.
Arm parameters (all units are in SI)

Parameters	Nominal values	Variation zone
Mass	$M = 2$	$1 \leqslant M \leqslant 5$
Joint stifness	$k = 100$	$k = 100$
Length ($2L$)	$L = 1$	$1 \leqslant L \leqslant 3$
Gravity	$g = 9.8$	$g = 9.8$
Inertia	$I = 1.5$	$1 \leqslant I \leqslant 2$
Motor inertia	$J = 1.5$	$1 \leqslant J \leqslant 2$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{-MgL}{I}\sin(x_1) - \dfrac{k}{I}(x_1 - x_3) \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \dfrac{k}{J}(x_1 - x_3) + \dfrac{1}{J}u \end{cases} \tag{60}$$

in which $x_1 = q_1$ and $x_3 = q_2$. In the limit of $k \longrightarrow \infty$ the rigid model of the system is given by:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-MgL}{(I+J)}\sin(x_1) + \frac{1}{(I+J)}u \end{aligned} \tag{61}$$

in which $x_1 = q_1 = q_2$. By choosing $q_1 = q$ and $z = k(q_1 - q_2)$ as the elastic force, the model of the system can be rewritten in a singular perturbation form:

$$\begin{cases} \ddot{q} = \dfrac{-MgL}{I}\sin(q) - \dfrac{1}{I}z \\ \epsilon\ddot{z} = \dfrac{-MgL}{I}\sin(q) - \left(\dfrac{1}{I} + \dfrac{1}{J}\right)z - \dfrac{1}{J}u \end{cases} \tag{62}$$

in which $\epsilon = 1/k$.

Spong has proposed a composite control law for this system in which there exists two control components corresponding to the fast and slow dynamics. The slow dynamic component is composed of a control law based on the rigid model of the system in addition to a corrective term, which is a feedback linearization algorithm based on the rigid model of the system [4]. According to the rigid model of the system given in (61), the feedback linearization control signal can be chosen as:

$$u_0 = (I+J)V + MgL\sin(x_1) \tag{63}$$

in which V is the linear component of it and can be given as:

$$V = \dot{x}_2^{\mathrm{d}} - a_1(x_1 - x_1^{\mathrm{d}}) - a_2(x_2 - x_2^{\mathrm{d}}). \tag{64}$$

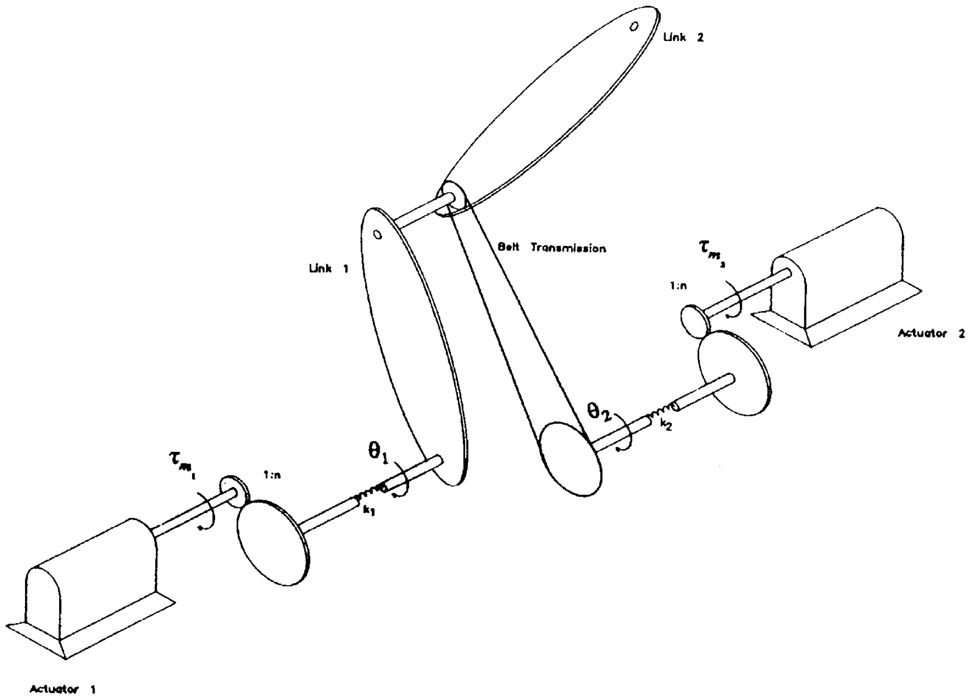

Figure 1. Two-link flexible joint manipulator.

In order to derive the corrective term, the integral manifold and the control law are expanded as follows:

$$H = H_0 + \epsilon H_1 + O(\epsilon^2)$$
$$u_s = u_0 + \epsilon u_1 + O(\epsilon^2)$$

substitute these relations into (62) and equating the similar terms, we have:

$$H_0 = \frac{-MgLJ}{I+J}\sin(q) - \frac{I}{I+J}u_0, \tag{65}$$

and, similarly:

$$u_1 = \ddot{H}_0. \tag{66}$$

Choose:

$$u_{\mathrm{f}} = 1\eta + 1\dot{\eta} \tag{67}$$

in which η corresponds to the variation of z from manifold H. Hence, the composite control law is given by:

$$u = u_{\mathrm{s}} + u_{\mathrm{f}} = u_0 + \epsilon u_1 + u_{\mathrm{f}} \tag{68}$$

in which u_0, u_1 and u_{f} are evaluated in (63), (66) and (67), respectively.

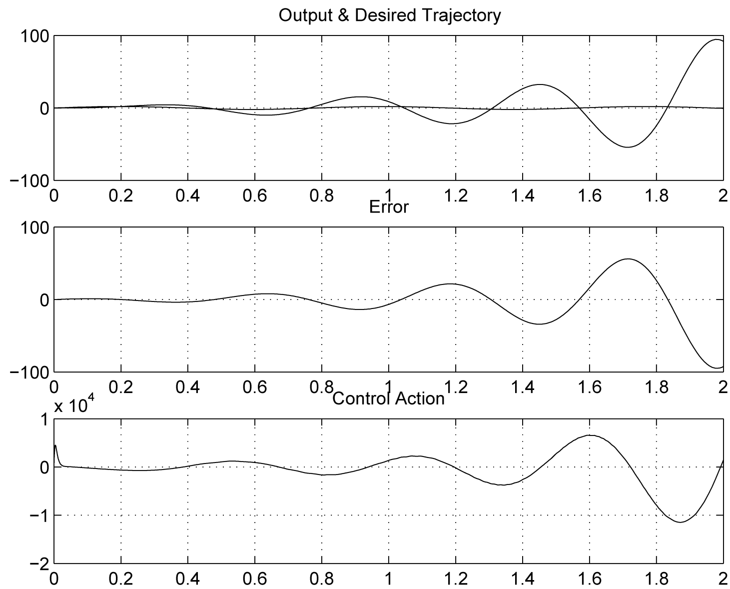

Figure 2. Instability of the closed-loop system by applying only the rigid controller term u_0; Spong algorithm.

As is illustrated in Fig. 2, the closed-loop system becomes unstable, provided that only the corresponding rigid control effort u_0 is applied on the system. However, as illustrated in Fig. 3 the system becomes stable and the desired trajectory $q_d = 2\sin(8t)$ is well tracked, implementing the proposed composite control on the nominal model of the system. However, this algorithm is not robust to the model parameter variations. As illustrated in Fig. 4, the tracking performance becomes quite poor for the maximum perturbation values for the parameters I, J, M and L.

For the sake of comparison, the proposed robust PID controller may be now applied on the same system. The proposed control law is composed of three terms as given in (68), in which the rigid control law is a PID controller whose coefficients satisfy the robust stability conditions elaborated in Theorem 4 as follows:

$$u_0 = 200\dot{e} + 500e + 100 \int_0^t e(s)\,\mathrm{d}s.$$

The integral manifold would be:

$$H_0 = -4.9\sin(q) - \frac{1}{2}u_0,$$

and the corrective term corresponds to:

$$u_1 = \ddot{H}_0.$$

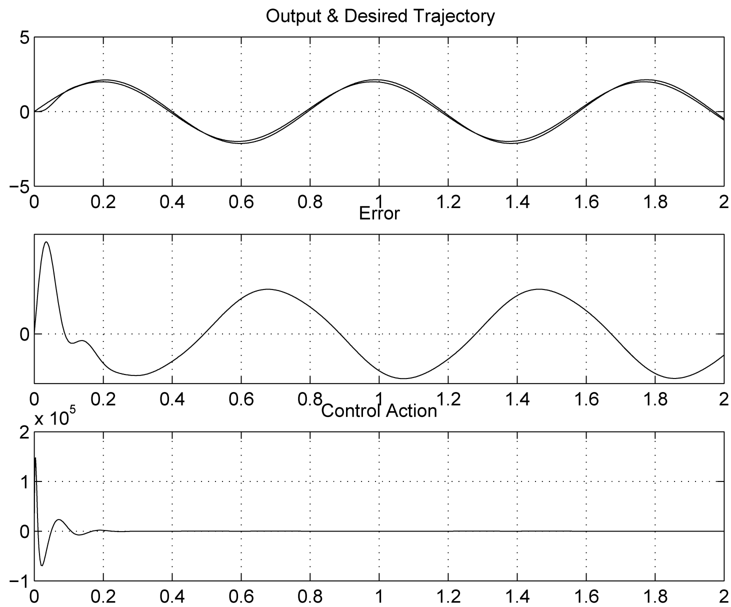

Figure 3. Tracking performance of the closed-loop system for the nominal model; Spong algorithm.

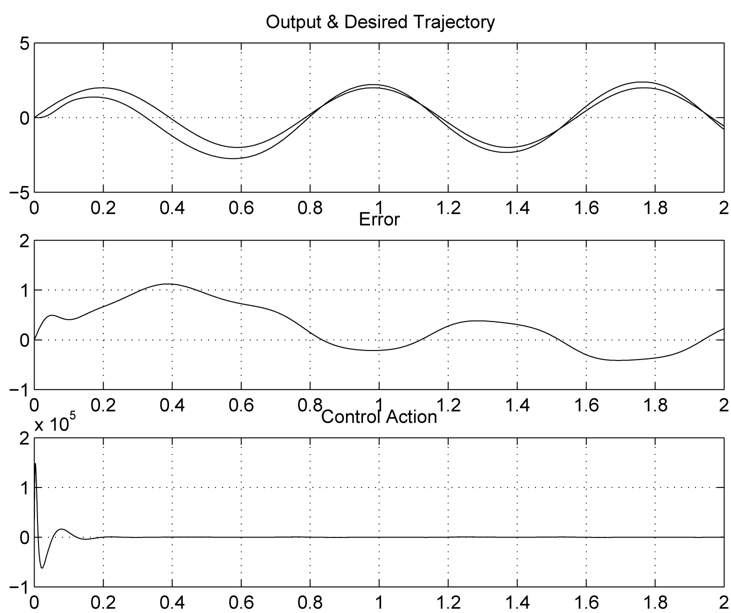

Figure 4. Poor tracking performance of the closed-loop system for the perturbed model; Spong algorithm.

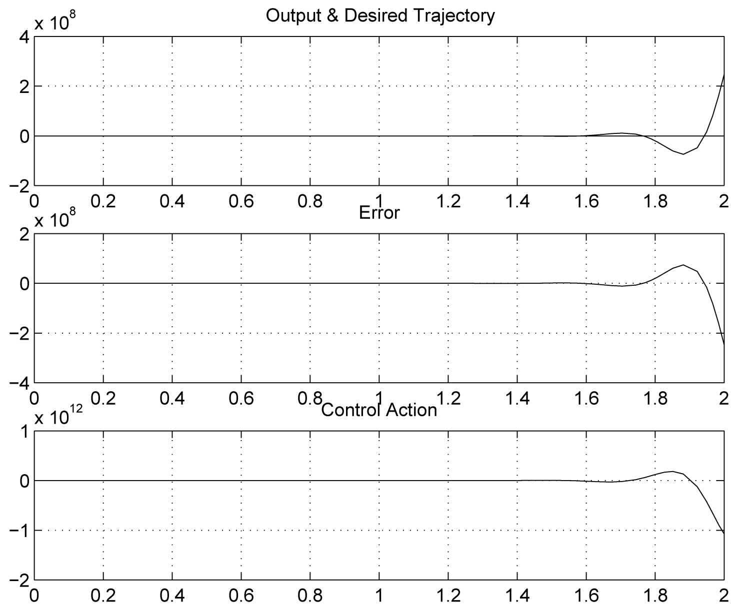

Figure 5. Instability of the closed-loop system by applying only the rigid controller term u_0; proposed algorithm.

The fast control law is a simple PD controller satisfying the robust stability conditions such as:

$$u_f = 5\eta + 5\dot{\eta}$$

in which η indicates the variation of z from the integral manifold H.

It was observed before that if only the rigid term of the composite control law is implemented on the closed-loop system, the system becomes unstable as illustrated in Fig. 5. However, by implementing the complete proposed control law, not only does the system track the desired trajectory for the nominal parameters of the model (Fig. 6), but also the robust stability and tracking performance of the system with maximum variation in its model parameters are preserved (Fig. 7).

The simulation results clearly show the effectiveness of the proposed control algorithm to robustly stabilize the system, while achieving robust performance. The superiority of our proposed algorithm compared to Spong's algorithm is its robustness to the model variations and the simplicity of its implementation. To quantitatively compare the tracking errors obtained by these methods, note that the two-norm of the tracking error in the Spong algorithm is 20.73, while its infinity-norm is about 1.123. Using our proposed method these values are reduced to 2.93, and 0.447, respectively, despite the simililarity in the norm of the actuator efforts. Hence, the proposed algorithm is not only robust to the model variation, but also improves the tracking performance quite significantly. In order to evaluate the effectiveness of the proposed method at high acceleration demands, a sinusoid reference trajectory a frequency of 20 rad/s is simulated in Fig. 8. As it is

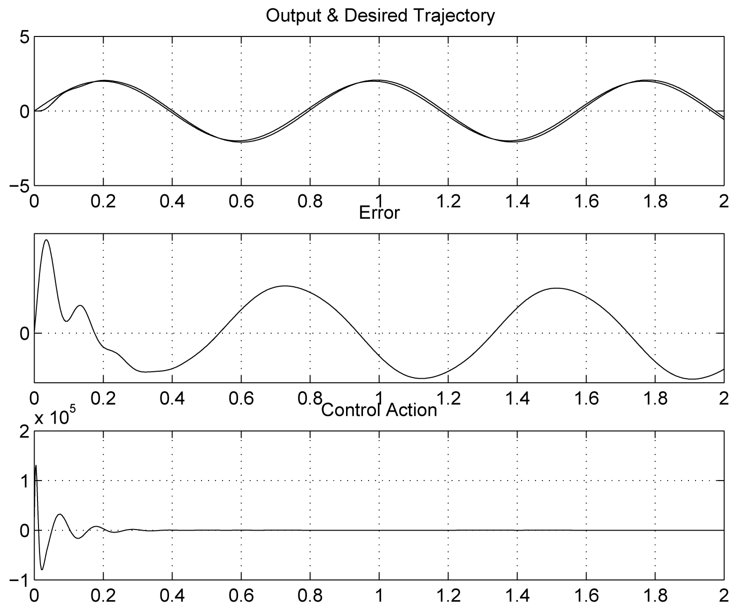

Figure 6. Tracking performance of the closed-loop system for the nominal model; proposed algorithm.

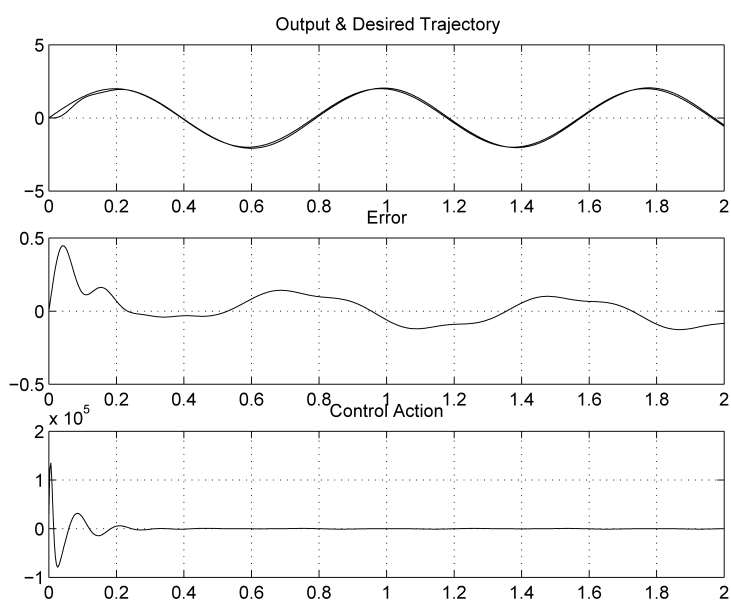

Figure 7. Suitable tracking performance of the closed-loop system for the perturbed model; proposed algorithm.

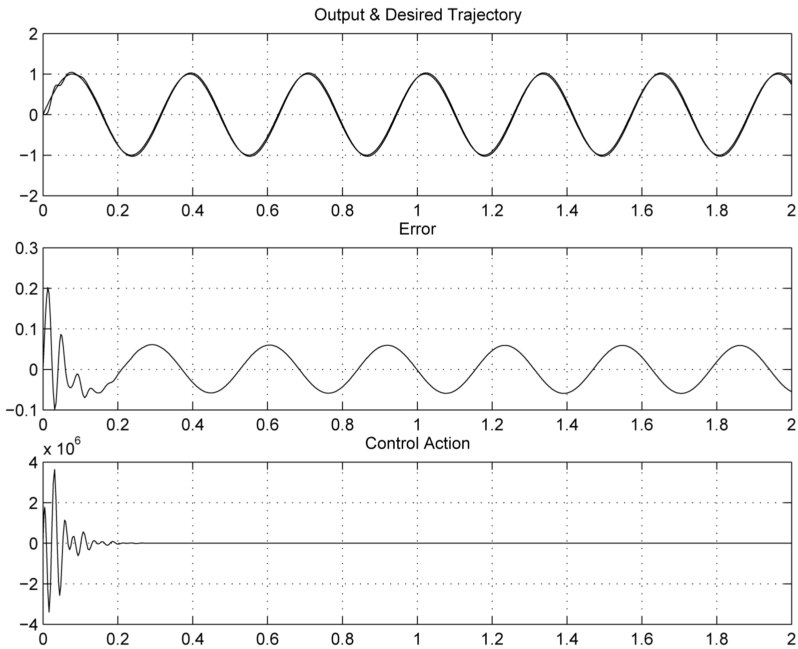

Figure 8. Suitable tracking performance of the closed-loop system for the perturbed model at high acceleration demand; proposed algorithm.

clearly illustrated, the tracking performance of the proposed controller is still quite desirable.

6.2. Multiple-link flexible joint manipulator

Consider the two-link flexible joint manipulator illustrated in Fig. 1. In this manipulator the joint flexibility is modeled with a linear torsional spring with stiffness k. The equation of motion of this system is [6]:

$$\begin{aligned} m_{11}\ddot{\theta}_1 + m_{12}\ddot{\theta}_2 + C_{21}\dot{\theta}_2^2 + G_1 + k_1(\theta_1 - \phi_1) = 0, \\ m_{21}\ddot{\theta}_1 + m_{22}\ddot{\theta}_2 + C_{12}\dot{\theta}_1^2 + G_2 + k_2(\theta_2 - \phi_2) = 0, \end{aligned}$$

$$N_1^2 J_{m1}\ddot{\phi}_1 - k_1(\theta_1 - \phi_1) = u_1, \tag{69}$$

$$N_2^2 J_{m2}\ddot{\phi}_2 - k_2(\theta_2 - \phi_2) = u_2$$

in which m_{ij} are the elements of the following mass matrix:

$$M(\theta, \phi) = \begin{bmatrix} m_1 l_{c1}^2 + m_2 l_1^2 + J_{l_1} & m_2 l_1 l_{c2} \cos(\phi_1 - \theta_1) \\ m_2 l_1 l_{c2} \cos(\phi_1 - \theta_1) & m_2 l_{c2}^2 + J_{l2} \end{bmatrix}, \tag{70}$$

m_i and J_{li} are the mass and the moment of inertia of the ith link, while l_i and L_{ci} are the link length and the distance of the center of mass of ith link to its joint, respectively. The other terms of (70) are given as:

$$C_{21} = -m_2 l_1 l_{c2} \sin(\phi_1 - \theta_1), \quad G_1 = (m_1 l_{c1} + m_2 l_1) g \cos(\theta_1), \tag{71}$$

$$C_{12} = m_2 l_1 l_{c2} \sin(\phi_1 - \theta_1), \quad G_2 = m_2 l_{c2} g \cos(\theta_2)$$

in which g is the gravity constant, k_i is the stiffness of the ith spring, J_i is the moment of inertia of the ith link and N_i is the ith gearbox ratio. The numerical parameters used for simulations are as follows [6]:

$$m_1 = m_2 = 1, \quad J_{l1} = J_{l2} = 1, \quad k_1 = k_2 = 100,$$
$$N_1^2 J_{m1} = N_2^2 J_{m2} = 1, \quad l_1 = l_2 = 1, \quad l_{c1} = l_{c2} = 0.5$$

Borrowing this system from Ref. [6], our proposed algorithm is applied to the system for comparison of the results. The equation of motion of the system can be reformulated in the standard form of a singular perturbation, using $\epsilon = 1/k_1 = 1/k_2 = 1/k = 0.01$ as the singular perturbation parameter. Defining two new state variables $z_1 = k(\theta_1 - \phi_1)$ and $z_2 = k(\theta_2 - \phi_2)$ as the elastic torques in the compliant elements, then:

$$\begin{bmatrix} 2.25 & 0.5\cos(\epsilon z_1) \\ 0.5\cos(\epsilon z_1) & 1.25 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 14.7\cos(\theta_1) + 0.5\dot{\theta}_2^2 \sin(\epsilon z_1) \\ 4.9\cos(\theta_2) - 0.5\dot{\theta}_1^2 \sin(\epsilon z_1) \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{72}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} - \epsilon \begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} - \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{73}$$

The corresponding rigid model when $k \longrightarrow \infty$ will be:

$$\begin{bmatrix} 3.25 & 0.5 \\ 0.5 & 2.25 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 14.7\cos(\theta_1) \\ 4.9\cos(\theta_2) \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{74}$$

As elaborated previously, the integral manifold for the corresponding system can be defined as:

$$z_1 = H_1\big(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, u_1, u_2, \epsilon\big); \quad z_2 = H_2\big(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2, u_1, u_2, \epsilon\big) \tag{75}$$

in which H_1 and H_2 satisfy the manifold condition. Expanding the manifolds up to first degree:

$$H_1 = H_1^0 + \epsilon H_1^1 + O\big(\epsilon^2\big); \quad H_2 = H_2^0 + \epsilon H_2^1 + O\big(\epsilon^2\big), \tag{76}$$

and expanding the corresponding control efforts as:

$$u_{1s} = u_1^0 + \epsilon u_1^1 + O\big(\epsilon^2\big), \quad u_{2s} = u_2^0 + \epsilon u_2^1 + O\big(\epsilon^2\big). \tag{77}$$

Hence, the reduced order model is evaluated as follows:

$$\begin{bmatrix} 2.25 & 0.5\cos(\epsilon H_1^0) \\ 0.5\cos(\epsilon H_1^0) & 1.25 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 14.7\cos(\theta_1) + 0.5\dot{\theta}_2^2\sin(\epsilon H_1^0) \\ 4.9\cos(\theta_2) - 0.5\dot{\theta}_1^2\sin(\epsilon H_1^0) \end{bmatrix}$$
$$+ \begin{bmatrix} H_1^0 + \epsilon H_1^1 \\ H_2^0 + \epsilon H_2^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{78}$$

In order to evaluate the fast dynamics caused by the joint flexibility, the normalized time variable $\tau = t/\sqrt{\epsilon}$ is used. Hence:

$$\frac{\mathrm{d}^2\eta_1}{\mathrm{d}\tau^2} = \epsilon\frac{\mathrm{d}^2\eta_1}{\mathrm{d}t^2} = -\eta_1 - u_1^{\mathrm{f}}(\eta_1, \eta_2),$$
$$\frac{\mathrm{d}^2\eta_2}{\mathrm{d}\tau^2} = \epsilon\frac{\mathrm{d}^2\eta_2}{\mathrm{d}t^2} = -\eta_2 - u_2^{\mathrm{f}}(\eta_1, \eta_2), \tag{79}$$

$$\eta_1 = z_1 - H_1^0, \quad \eta_2 = z_2 - H_2^0$$

in which H_1^0 and H_2^0 can be evaluated simply by replacing $\epsilon = 0$ in (73):

$$H_1^0 = \ddot{\theta}_1 - u_1^0, \quad H_2^0 = \ddot{\theta}_2 - u_2^0. \tag{80}$$

In order to evaluate the integral manifold and the control law for this system, (73) is used, substituting $z_i = H_i$ and equating up to first-order term with respect to ϵ. This concludes to:

$$H_1^1 = -\ddot{H}_1^0 - u_1^1, \quad H_2^1 = -\ddot{H}_2^0 - u_2^1. \tag{81}$$

Expanding (80) to the first order of ϵ we have:

$$H_1^1 = -0.5\dot{\theta}_2^2 H_1^0, \quad H_2^1 = 0.5\dot{\theta}_1^2 H_1^0, \tag{82}$$

and from (81) we get:

$$u_1^1 = 0.5\dot{\theta}_2^2 H_1^0 - \ddot{H}_1^0, \quad u_2^1 = -0.5\dot{\theta}_1^2 H_1^0 - \ddot{H}_2^0. \tag{83}$$

Finally, the slow part of the control law will be calculated from:

$$u_{1\mathrm{s}} = u_1^0 + \epsilon u_1^1, \quad u_{2\mathrm{s}} = u_2^0 + \epsilon u_2^1. \tag{84}$$

u_1^0 and u_2^0 are the rigid part of the control law, and as elaborated before are robustly designed as a PID controller. In here we design the PID gains which satisfy the robust conditions:

$$u_1^0 = 500e + 50\dot{e} + 50\int_0^t e(s)\,\mathrm{d}s, \tag{85}$$
$$u_2^0 = 200e + 50\dot{e} + 50\int_0^t e(s)\,\mathrm{d}s.$$

The fast control law is also designed as a PD controller as:

$$u_{1\mathrm{f}} = \eta_1 + \dot{\eta}_1, \quad u_{2\mathrm{f}} = \eta_2 + \dot{\eta}_2. \tag{86}$$

Finally, the control law is composed of the fast and slow parts:

$$u_1 = u_{1\mathrm{s}} + u_{1\mathrm{f}}, \quad u_2 = u_{2\mathrm{s}} + u_{2\mathrm{f}}. \tag{87}$$

To compare the simulation results to [6], the reference signal is considered as:

$$\theta_i = 1.57 + 7.8539 \exp(-t) - 9.428 \exp(-t/1.2) \quad i = 1, 2 \tag{88}$$

in which the joint angles reach a final value of $\theta_i = \pi/2$ from the zero state. As shown in Fig. 9, the system reaches instability if only the rigid control is applied to the system. The main reason for instability is the divergence of its fast dynamics.

Figure 10 illustrates the respones of the system to our proposed composite control law. The system becomes stable and the tracking performance is quite desirable, despite the limited control effort guaranteed with adding a saturation block in the simulation (Fig. 11). In order to analyze the robustness of the response, the system parameters were varied by 50%. Figures 12 and 13 illustrate the robustness of the performance and stability to the model variations.

In order to compare the effectiveness of our proposed control law, the simulation results are compared to the results presented in Ref. [6]. Al-Ashoor *et al.* have used a robust-adaptive control law in addition to the composite law we introduced in this paper. By this means, in addition to the corrective adaptive term used based on the integral manifold, another term is used for robustness of the performance against the modeling uncertainties. Figure 14 illustrates the results obtained for the refrence signal introduced in equation (88) in Ref. [6]. Figure 14 illustrates

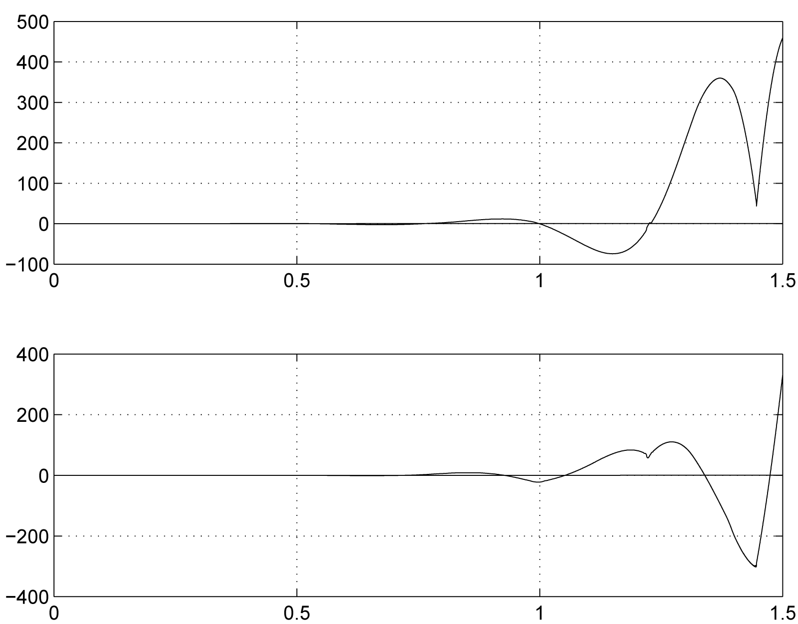

Figure 9. Instability of the closed-loop system by applying only the rigid controller term u_0; proposed algorithm.

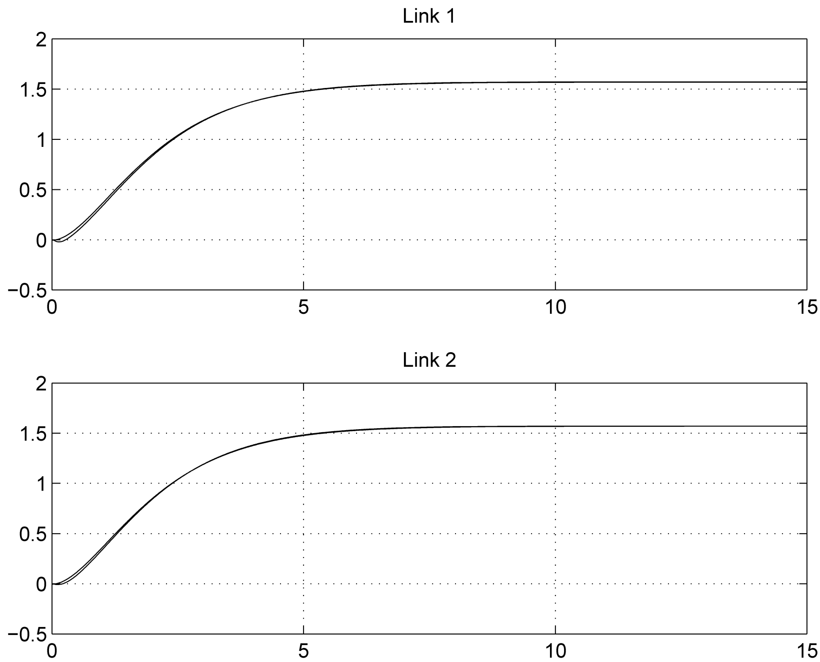

Figure 10. Tracking performance of the closed-loop system for the nominal model; proposed algorithm.

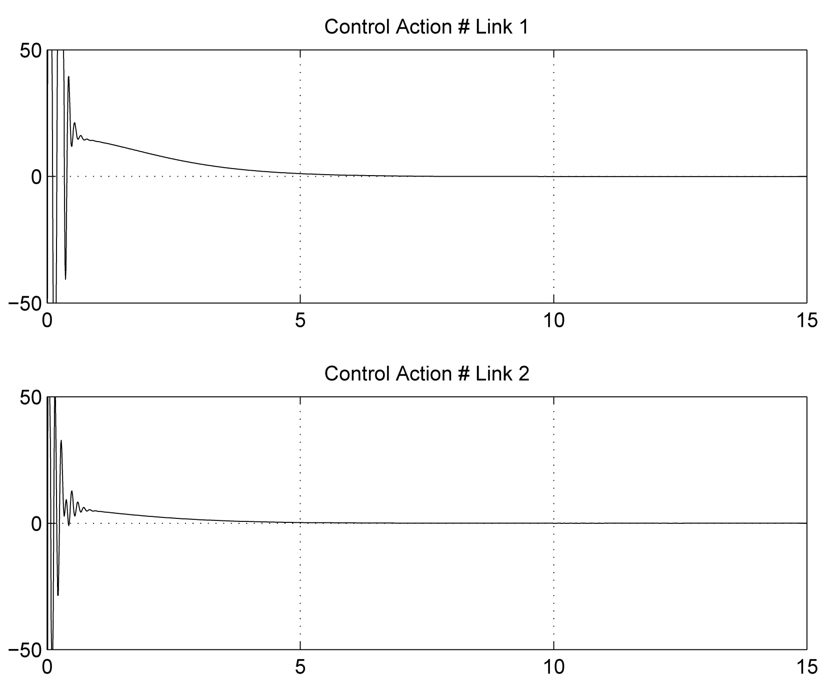

Figure 11. Control effort for the closed-loop system and the nominal model; proposed algorithm.

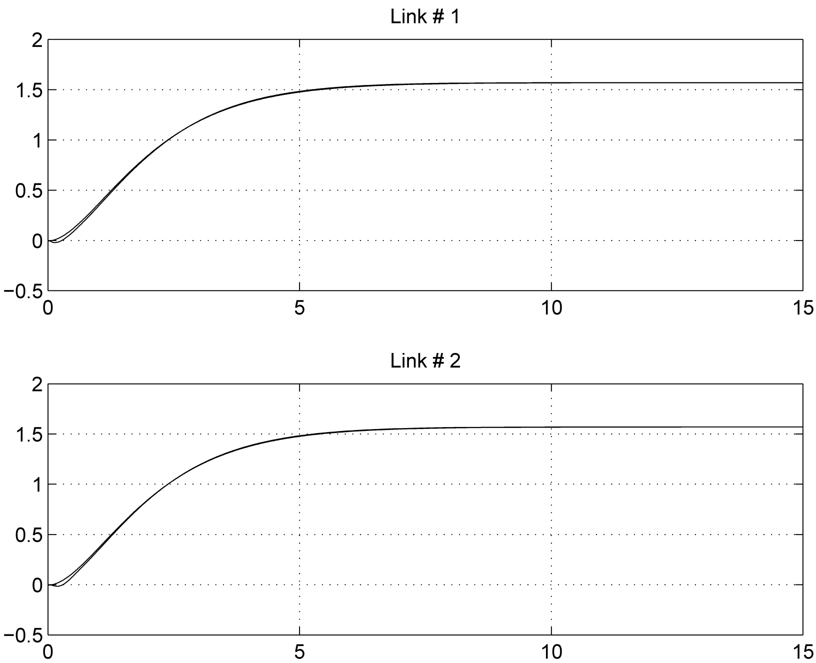

Figure 12. Tracking performance of the closed-loop system for the perturbed model; proposed algorithm.

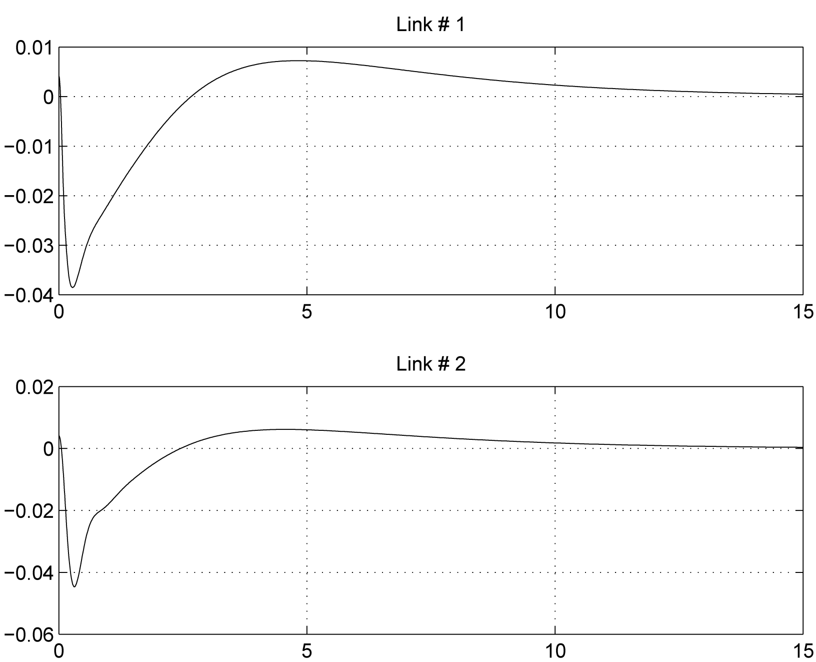

Figure 13. Tracking error for the closed-loop system and the perturbed model; proposed algorithm.

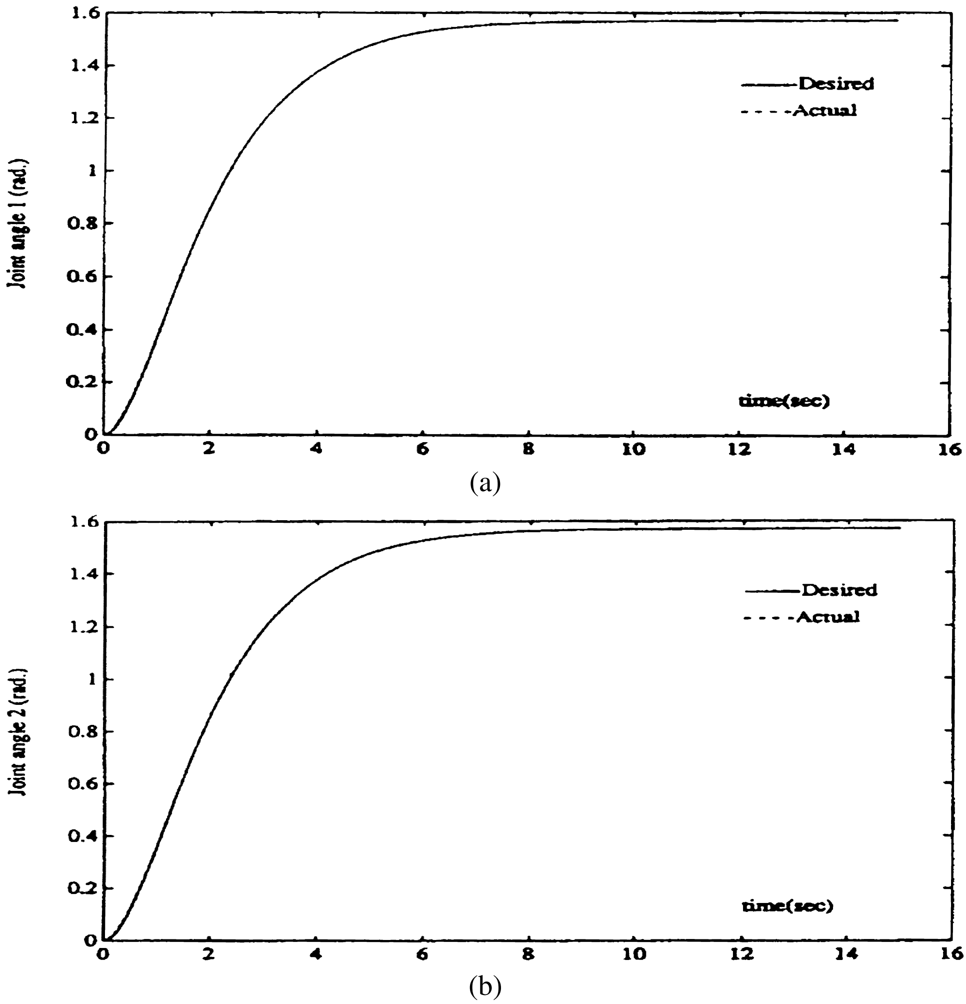

Figure 14. Tracking performance of the closed-loop system for the nominal model; Al-Ashoor *et al.* algorithm.

the tracking performance despite the bounded control effort illustrated in Fig. 15. Comparing these results to that obtained with our proposed control law (Figs 10 and 11) it is clear that, despite the simplicity of our proposed control law, the results are quite similar. Hence, our proposed algorithm results into a much simpler implementation effort without loss of performance. The only limitation that exists in our proposed law compared to that in Ref. [6] is the amplitude of the control law at the initial time of the simulation. The adaptive law has a smaller control effort in the beginning of the simulation, due to the adaptive nature of the algorithm and using the information of the identified model of the system in the control law. This issue is under current investigation, and promising results are obtained either by an H_∞-based robust performance synthesis for PID design [21], or adapting the

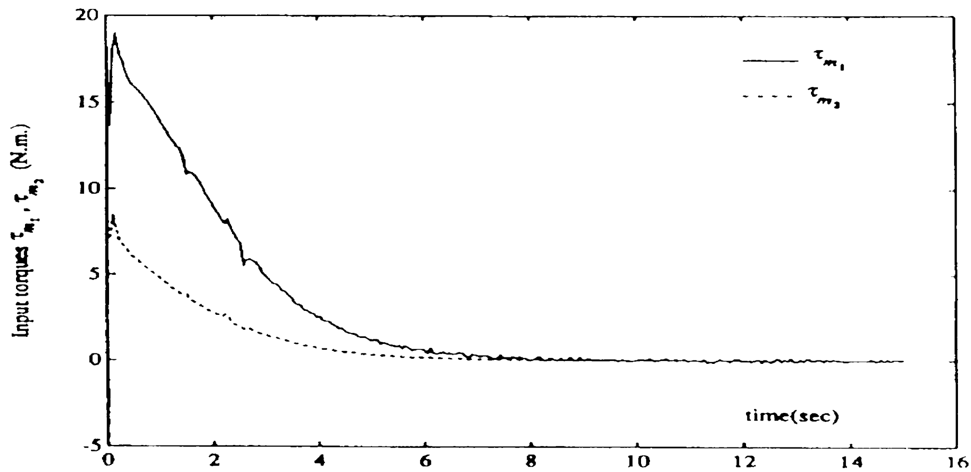

Figure 15. Control effort for the closed-loop system and the nominal model; Al-Ashoor *et al.* algorithm.

controller gain by a supervisory control loop [22–24]; in both cases the control effort can be limited to the desired bounds.

7. CONCLUSIONS

The control of flexible joint manipulators is examined in detail. First, the model of N-axis robot manipulators is given and reformulated in the form of a singular perturbation, and an integral manifold is used to separate fast and slow dynamics. A composite control algorithm is proposed in order to achieve the required performance, consisting of a corresponding control law for fast and slow subsystems. A simple PD control is proposed for the fast subsystem, and it is proven that the fast subsystem becomes asymptotically stable and the flexible manifold becomes invariant. The slow subsystem itself is controlled through a robust PID control, which is designed based on the rigid model, and a correction term, whose design is based on the reduced flexible model. The robust UUB stability of the PID controller is first analyzed by Lyapunov theory. Then, the stability of the complete closed-loop system is analyzed and the detailed stability conditions are derived for the closed-loop system. It is shown that by choosing proper gains for the proposed controller, robust stability of the closed-loop system is guaranteed despite the modeling uncertainties. Finally, the effectiveness of the proposed control law is verified through simulations. Single- and two-link flexible joint manipulators are examined in this study, and the simulation results are compared to that given in the literature. The effectiveness of preserving the robust stability and obtaining desirable performance for the closed loop system is verified and compared, respectively.

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