Forward Kinematic Problem and Constant Orientation Workspace of 5-RPRRR (3T2R) Parallel Mechanisms

Mehdi Tale Masouleh, Clément Gosselin
Department of Mechanical Engineering
Laval University
Quebec, QC, Canada, G1V 0A6
mehdi.tale-masouleh.1@ulaval.ca,
gosselin@gmc.ulaval.ca

Mohammad Hossein Saadatzi, Hamid D. Taghirad
Advanced Robotics & Automated Systems (ARAS),
Faculty of Electrical & Computer Engineering
K.N. Toosi University of Technology
Tehran, Iran
saadatzi@ee.kntu.ac.ir, Taghirad@kntu.ac.ir

***Abstract*— This paper investigates some kinematic properties of a five-degree-of-freedom parallel mechanism generating the 3T2R motion and comprising five identical limbs of the RPUR type. In this study, two classes of simplified designs are proposed whose forward kinematic problems have either a univariate or a closed-form solution. The principal contributions of this study are the solution of the forward kinematic problem for some simplified designs— which may have more solutions than the FKP of the general 6–DOF Stewart platform with 40 solutions— and the determination of the constant orientation workspace based on algebraic geometry (Bohemian domes).**

Keywords- 5–DOF parallel mechanisms; Forward kinematic problem; Constant orientation workspace; Bohemian dome

I. Introduction

Five–degree–of–freedom (DOF) parallel mechanisms are a class of parallel mechanisms with reduced DOFs which, according to their mobility, fall into three classes: (1) three translational and two rotational freedoms (3T2R), (2) three rotational and two planar translational freedoms (3R2Tp) and (3) three rotational and two spherical translational freedoms (3R2Ts) [9]. Since, in the industrial context, the 3T2R motion can cover a wide range of applications including, among others, 5-axis machine tools and welding, therefore, in this research, the kinematic properties of this class will be investigated. In medical applications that require at the same time mobility, compactness and accuracy around a functional point, 5-DOF parallel mechanisms can be regarded as a very promising solution [1].

Although hexapods, 6–DOF parallel mechanisms, can be used as versatile robots and machine tools, their complexity is a major deterrent to their widespread in industry [11] which stimulates interest for parallel mechanisms with lower-mobility in some particular applications. It is generally believed that in comparison with a general-purpose manipulator a limited-DOF parallel manipulator has the advantages of simple mechanical structure, low manufacturing cost, simple control algorithm, and therefore high-speed capability [2].

As far as 5–DOF mechanisms with identical limb structures are concerned, researchers have mainly worked on the type synthesis [9, 3–6]. In fact, it was believed that symmetrical[1] 5-DOF mechanisms could not be built [13] until [14] proposed a first architecture.

To the best knowledge of the authors, up to now, very few kinematic studies have been conducted on symmetrical 5-DOF parallel mechanisms (especially in 3T2R symmetrical parallel mechanisms). This is probably due to their short history.

The main focus of this research is the FKP of symmetrical 5-DOF parallel mechanisms, more precisely 5-RPRRR, which can be regarded as one of the most challenging topics in the kinematics of parallel mechanisms. The analytical resolution of the FKP in the context of parallel mechanisms, due to its mathematical complexities, initiated several researches. In some cases, upon considering design conditions, such as the coalescence of connection points and planar base and platform, the FKP can be expressed in a closed–form solution, i.e., an explicit solution for the FKP. The general approach toward obtaining a univariate expression for the FKP is based on *elimination theory*, such as the *Resultant* method.

The remainder of this paper is organized as follows. The architecture and the general kinematic properties of the 5-RPRRR parallel mechanism which originated from the type synthesis performed in [6, 9] are first outlined. The FKP are addressed and from the results obtained two classes of simplified designs— which include in total 9 simplified designs— are found whose FKP have either a univariate or a closed–form solution. The constant–orientation workspace is interpreted geometrically and the results are implemented in a CAD

The authors would like to acknowledge the financial support of the Natural Sciences and Engineering Research Council of Canada (NSERC) as well as the Canada Research Chair program.

Proceedings of ICEE 2010, May 11-13, 2010
978-1-4244-6761-7/10/$26.00 ©2010 IEEE

[1] In the context of this paper, the symmetric properties refer to the limb type and not to the geometry, such as centro–symmetrical simplifications

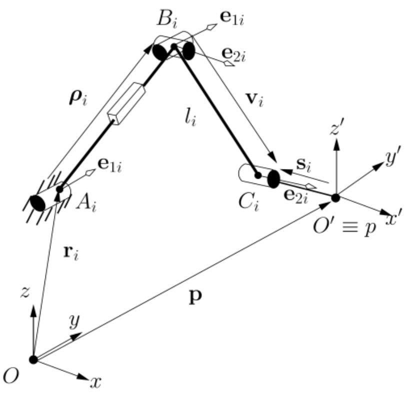

Figure 2. Schematic representation of a RPUR limb.

system. Moreover, an algorithm, inspired from the one presented in [7] is proposed for computing the boundary and volume of the constant orientation workspace.

II. Architecture and Kinematic Modeling

Figs. 1 and 2 provide respectively a representation of a RPUR limb and a CAD model for a 5-DOF parallel mechanism providing all three translational DOFs, plus two independent rotational DOFs (3T2R) of the end-effector, namely (x, y, z, φ, θ) In the latter notation, (x, y, z) are the components of vector p and represent the translational DOFs with respect of the fixed frame O, illustrated in Fig. 1, and (φ, θ) stand for the orientation DOFs around axes x and y($\mathbf{e}_{1i}$), respectively. The rotation from the fixed frame O_{xyz} to the moving frame $O_{x'y'z'}$ is defined as follows: a first rotation of angle φ is performed around x−axis followed by the second rotation about $\mathbf{e}_{1i}$ by angle θ. For more information concerning the kinematic modeling see [11].

From [11] it follows that Q cannot be prescribed arbitrarily since the mechanism has only two degrees of rotational freedom. Therefore, a rotation matrix consistent with the orientation capabilities of the mechanism must be chosen. Indeed, the motion capabilities of the mobile platform should be limited to the position and orientation of a line attached to the mobile platform. Hence, based on the results presented in [12] and on the definition of angles φ and θ given above, this rotation matrix can be written as:

$$Q = \begin{bmatrix} cos\theta & sin\varphi sin\theta & cos\varphi sin\theta \\ 0 & cos\varphi & -sin\varphi \\ -sin\theta & sin\varphi cos\theta & cos\varphi cos\theta \end{bmatrix}. \tag{1}$$

III. Forward Kinematic Problem (FKP)

The FKP pertains to finding the pose of the platform for a given set of actuated joints. With reference to Fig. 1, the following equations, arising from the kinematic constraint of the i_{th} limb, can be written:

$$(x_{Bi} - x_{Ai})^2 + (z_{Bi} - z_{Ai})^2 = \rho_i^{\,2} \tag{2}$$

$$(x_{Ci} - x_{Bi})^2 + (y_{Ci} - y_{Bi})^2 + (z_{Ci} - z_{Bi})^2 = l_i^2 \tag{3}$$

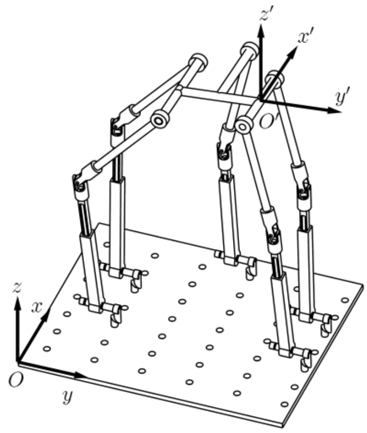

Figure 1. CAD model of a 5-RPUR parallel mechanism.

$$(x_{Ci} - x_{Bi})cos\theta - (z_{Ci} - z_{Bi})sin\theta = 0 \tag{4}$$

Such that the first two equations represent, respectively, the magnitude of $\boldsymbol{\rho}_i$ and $\mathbf{v}_i$ and the last one corresponds to the kinematic constraints between $\mathbf{e}_2$ and $\mathbf{v}_i$, i.e., $\mathbf{e}_2 \perp \mathbf{v}_i$.

For the FKP, the above system of equations should be solved for (*x, y, z,* φ, θ) with respect to input data which are the lengths of the prismatic actuators, ρ_i.

Equations (2–4) contain the coordinates of the two passive joints, namely B_i and C_i. Since C_i is attached to the platform, its coordinates can be written directly in terms of the platform pose. One has:

$$[x_{Ci}, y_{Ci}, z_{Ci}]^T = p + Q\mathbf{s}_\mathbf{i}' . \tag{5}$$

Upon eliminating the coordinates of the passive variable B_i (x_{Bi}, y_{Bi}, z_{Bi}), having in mind that $y_{Bi} = y_{Ai}$, Equations (2–4) lead to (*x, y, z, u, t*) where *u* and *t* stand respectively for the *tan of half angle* substitution of φ and θ. The degrees of the equations are respectively (4,4,4,8,8). Thus it follows that the univariate expression in the three dimensional kinematic space for a RPUR limb is of degree 20. Numerous approaches were proposed in the literature and practice including the use of numerical procedures, simplifying the mechanism by the coalescence of some of the connection–points on the platform or the base and, finally, to use some extra sensors. In this project, simplifying the mechanism by the coalescence of some of the connection–points is considered for solving the FKP with the aim of obtaining a simpler design, reducing the mechanical interferences and increasing the workspace volume.

From the results obtained in [12], the following conclusion can be drawn: *Any mechanical simplification which provides the coordinates of two pairs of U joints explicitly or a relation among them leads to a univariate solution for the FKP.* The above issue remains central to the development of the simplified designs having either a univariate or a closed–form solution to the FKP. With the above conclusion in mind, consider two limbs, *i* and *j*, for which:

1. The connection points at the base, A_i and A_j, are in a plane with $\mathbf{e}_1$ as normal or coincide;

2. Both second moving links have the same length, $l_i = l_j$, or coincide;

3. The connection points on the platform, C_i and C_j, are aligned with $\mathbf{e}_2$, or coincide.

Therefore, in a design for which two pairs of limbs fulfill the latter conditions, on the basis of the above conclusion, FKP admits a univariate solution. There are three distinct situations, $\mathtt{S} = \{\mathtt{A}_1\mathtt{A}_2\mathtt{A}_3\}$, in which the latter conditions described above can occur as depicted in Fig. 3. Therefore, all second order subsets of $\mathtt{S}$ adopt a polynomial form for their FKP, namely:

$$\{\{\mathtt{A}_1\mathtt{A}_1\},\{\mathtt{A}_1\mathtt{A}_2\},\{\mathtt{A}_1\mathtt{A}_3\}\},$$
$$\{\{\mathtt{A}_2\mathtt{A}_3\},\{\mathtt{A}_2\mathtt{A}_3\},\{\mathtt{A}_3\mathtt{A}_3\}\}. \quad (6)$$

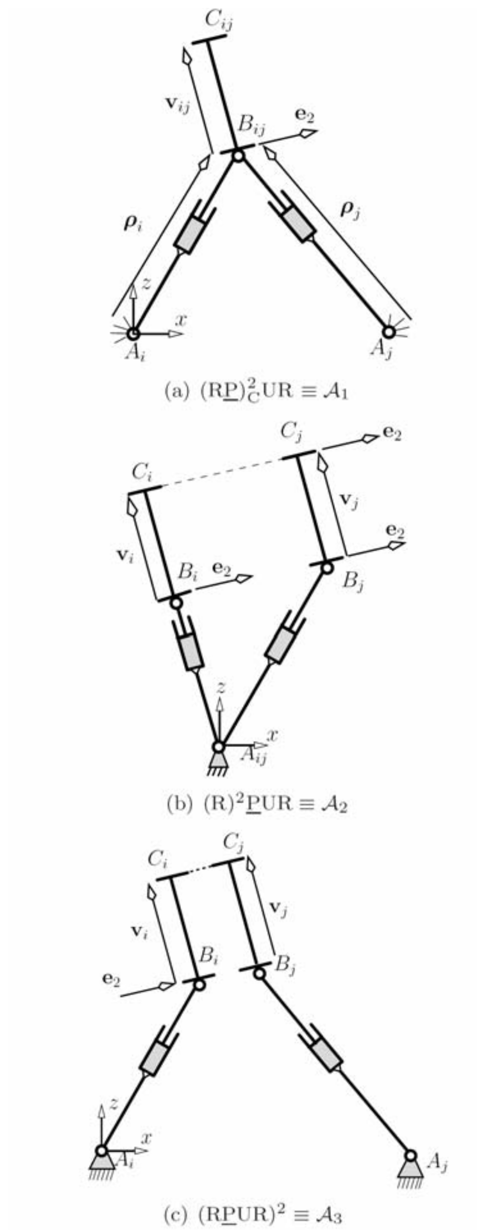

Figure 3. Simplified kinematic arrangements.

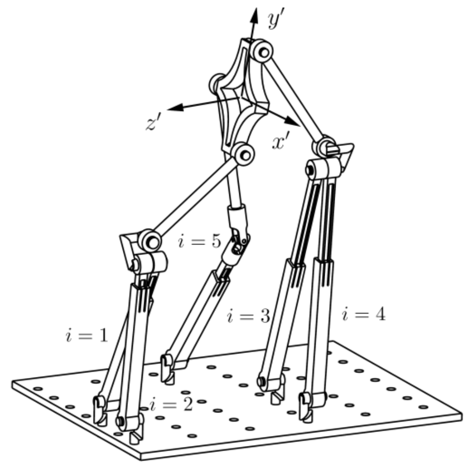

Figure 4. CAD model of a $\{\mathtt{A}_1\mathtt{A}_1\}$ parallel mechanism.

IV. CLOSED–FORM SOLUTION FOR THE FKP OF A $\{\mathtt{A}_1\mathtt{A}_1\}$ DESIGN

Fig. 4 represents a CAD model for a $\{\mathtt{A}_1\mathtt{A}_1\}$ design. Referring to Fig. 3(a), the coordinates of the U joints belonging to the simplified arrangement $\mathtt{A}_1$, B_{12} and B_{34}, can be readily computed and consist of the intersection of two circles centered at A_1 and A_2 (A_3 and A_4) with radius ρ_1 and ρ_2 (ρ_3 and ρ_4). Four solutions can be found as a whole for the coordinates of the latter U joints. Having in place the coordinates of these two joints and upon subtracting (4) for i=1, 2 from i = 3, 4 leads to:

$$(x_{B34} - x_{B12})\cos\theta - (z_{B34} - z_{B12})\sin\theta = (s_{34} - s_{12}). \quad (7)$$

Applying the half–tan substitution for t=tan(φ/2) results in a quadratic expression:

$$t = \frac{z_{B12} - z_{B34} \pm \sqrt{H}}{x_{B34} - x_{B12} + s_{34} - s_{12}} \quad (8)$$

With

$$H = (z_{B34} - z_{B12})^2 - (x_{B34} - x_{B12})^2 + (s_{34} - s_{12})^2. \quad (9)$$

From the above it can be deduced that q have up to 2×4=8 solutions (2 and 4 stand respectively for the quadratic expression and for the number of solution for the coordinates of the two U joints).

Having determined the value of θ and the coordinates of both U joints, the next step consists in computing the coordinates of the U joint belonging to the regular limb, B_5. Skipping mathematical derivations, (2) for i = 5 can be re-written with respect to the obtained values and solved for x_{B5} as follows:

$$x_{B5} = x_{A5}\sin^2\theta + \ell_5\cos\theta \pm \sin\theta\sqrt{\rho_5^2 - (x_{A5}\cos\theta - \ell_5)^2} \quad (10)$$

Where

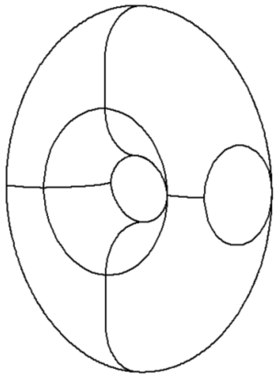

Figure 5. The constant orientation workspace for a RPUR limb by considering ρ_{min} = 250mm, ρ_{max} = 400mm, l = 150mm and θ = 0.

$$\ell_5 = x_{B1}\cos\theta - z_{B1}\sin\theta + (s_5 - s_1). \qquad (11)$$

From the above it can be concluded that two sets of solutions can be found for (x_{B5}, z_{B5}).

Reaching this step, all the passive variables, B_i, and θ are known. Combining (3) and (4) for the limbs i = 1(2),3(4),5 leads to:

$$\begin{cases} R_{12} = \kappa_{12}\cos^2\theta + (z_{C12} - z_{B12})^2 = 0 \\ R_{34} = \kappa_{34}\cos^2\theta + (z_{C34} - z_{B34})^2 = 0 \\ R_5 = \kappa_5\cos^2\theta + (z_{C5} - z_{B5})^2 = 0 \end{cases} \qquad (12)$$

This implies that the univariate expression for u=tan(φ/2), R_{u1}, is of degree six. Then, a back-solving procedure for (12) leads to construct the corresponding position of the mechanism (*x, y, z*) for a given (φ, θ). Consequently, the FKP of this mechanism admits up to 6×8×2 = 96 solutions where (6, 8, 2) is coming respectively from the upper bound of solutions for (R_{u1}, θ, B_5).

One could arrive to the same result for the upper bound of the FKP solutions upon considering a geometrical approach. Starting from (8), it follows that θ admits up to 8 solutions for the FKP. From Fig. 4, it can be seen that in a {A_1A_1} design the loop $B_{12}C_{12}C_{34}B_{34}$ can be made equivalent to a 4–bar linkage. This result is valid for all {A_iA_j} designs. As it is well–known, the motion of a 4–bar linkage generates a *sextic*, i.e., a sixth order curve [29]. Thus, in such a design, the FKP corresponds to the intersection of the sextic and a circle centered at B_5 which is generated by the regular limbs. From Bezout's theorem, it follows that this intersection results in 2×6 = 12 intersection points including two circular imaginary points as triple points [10]. Thus the intersection of the sextic and the circle results in up to 2×6–2×3 = 6 real intersection points (2 stands for the degree of the circle, 6 for the sextic and 3 for the imaginary points). From the IKP, it is known that there are two possibilities (two working modes) for the position of B_5. Taking account all the above factors, for one given value of θ the FKP of this mechanism results in 6×2 = 12 solutions. Since the 4–bar linkages can be constructed upon 8 ways then the upper bound for the number of postures of the FKP is 12×8 = 96 which is consistent with the conclusion reached above by direct manipulation of the equations.

TABLE I. GEOMETRIC PROPERTIES (IN MM) ASSUMED FOR A GENERAL 5-RPUR.

i	$(r_i)_x$	$(r_i)_y$	$(r_i)_z$	$(s'_i)_x$	$(s'_i)_y$	$(s'_i)_z$
1	-55	30	50	-50	0	0
2	245	30	50	50	0	0
3	20	205	0	0	50	0
4	200	180	0	0	50	-50
5	0	0	0	0	50	-50

V. WORKSPACE DETERMINATION

The complete workspace of the 5-RPUR manipulator can be regarded as a five–dimensional space for which no visualization exists. Geometrically, the problem of determining the constant–orientation workspace for a limb of the 5-RPUR parallel mechanism can be regarded as follows: For a fixed elongation of the prismatic actuator, the first revolute joint provides a circular trajectory centered at A_i with ρ_i as radius. The second link generates a surface by sweeping a second circle, with $\mathbf{e}_2$ as axis, along the first circle. Since the direction of $\mathbf{e}_2$ is prescribed and must remain constant, such a surface is a quadratic surface and is called a *Bohemian dome*. The above geometrical interpretation for the limb workspace of a RPUR is fully explained in [12]. A CAD model for the constant orientation of a RPUR limb can be found, as shown in Fig. 5, and is referred to here as B_i. Up to now, the geometry of the mobile platform has not been considered. Following the same method as in [7] the workspace of a limb attached to a platform can be computed by applying an offset vector -s_i to the limb workspace, B_i. Finally, the workspace of the mechanism is found by intersecting five offset B_i. Fig. 6, obtained with a CAD system, represents an example for the constant orientation workspace of a 5-RPUR parallel mechanism, whose design parameters are presented in TABLE I. In this section, it is assumed that l_i = 150mm, ρ_{min} = 250mm and ρ_{max} = 400mm. It should be noted that the mechanical interferences are omitted in this study. In what follows, an approach, inspired from the algorithm proposed in [7], is proposed which brings insight into

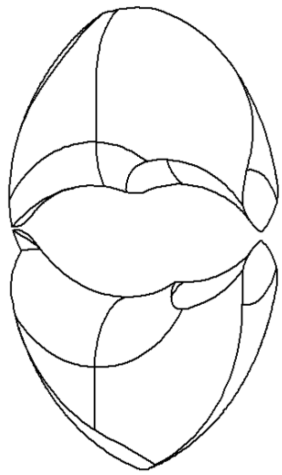

Figure 6. Constant orientation workspace for φ = 0 and θ = 0.

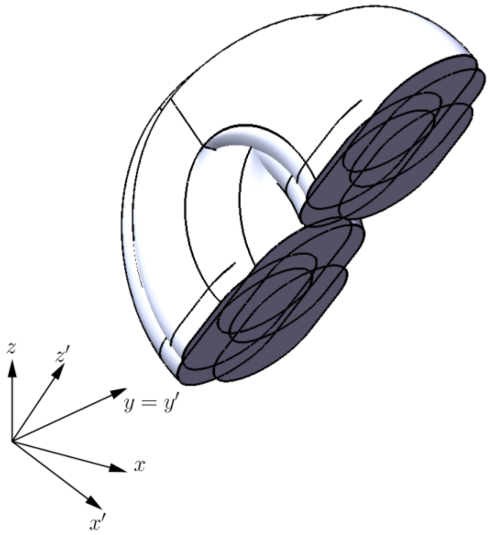

Figure 7. Cross sectioning five B_i by the plane H_i at x'_H.

the problem and is very useful during the design stage [8].

Being aware that each vertex space B_i should be offset by $-s_i$, then, mathematically, the center of each B_i can be expressed as follows:

$$w_i = r_i - Qs'_i. \tag{13}$$

Since in the case of the constant–orientation workspace we are dealing with a three dimensional space, a cross sectional plane should be considered in order to reduce the problem to a two dimensional one. From a geometrical inspection, it follows that a cross sectional plane, called H_i, which is rotated around the y axis of the fixed frame by angle θ results in a homogeneous section for the B_i and to conventional geometric objects such as circles and lines. Fig. 7 represents five intersected B_i for φ=π/4 and θ=π/4 for which H_i crosses at x'_H. This particular cross section implies that (13) should be multiplied by $Q^{-1}_{y,\theta}$ where $Q_{y,\theta}$ is the rotation around the y axis by angle θ:

$$w'_i = Q^{-1}_{y,\theta} w_i = Q^{-1}_{y,\theta} r_i - Q^{-1}_{x,\varphi} s'_i, \tag{14}$$

The next step is to determine the interval for which the

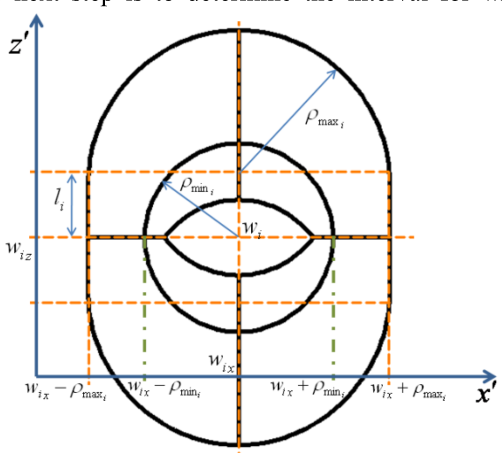

Figure 8. A schematic representation of a B_i including the used parameters.

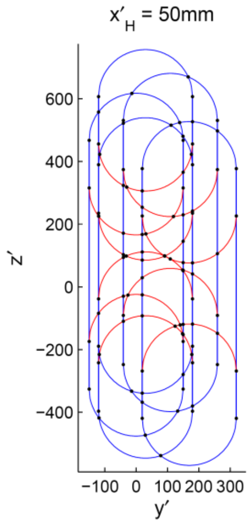

Figure 9. Cross section at x'_H = 50mm of five Bi and the possible intersections among them.

cross sectional plan should be applied in order to avoid non–essential cross sections. This can be done by considering Fig. 8 which represents schematically a *Bi* for a x'–z' view. From Fig. 8, it can be seen that H_i crosses all the B_is iff it lies inside of this interval:

$$\max(w'_{ix} - \rho_{\max i}) \le x'_H \le \min(w'_{ix} + \rho_{\max i}), \quad i = 1,2,\ldots 5 \tag{15}$$

Where

$$w'_i = [\, w'_{ix}, w'_{iy}, w'_{iz} \,]^T \tag{16}$$

As mentioned previously, the defined cross sectional plane, H_i, results in a homogeneous section for which, the obtained circles in each section have the same radius. Reaching this step, the problem of obtaining the constant orientation workspace of a 5-RPUR parallel mechanism is made equivalent to the determination of the constant orientation of the 6–DOF Gough–Stewart platform [7].

Fig. 9 shows a cross section at x'_H = 50mm for a design whose geometrical parameters are presented in TABLE I, for φ=π/4 and θ=π/4. From the number of possible point intersections, it can be deduced that the arrangement of these intersections in order to identify which ones constitute the boundary of the workspace should be a delicate task. The last step consists in obtaining all the circular arcs and lines defined by the intersection points found above by ordering these points.

This should be accompanied by a checking procedure to identify the arcs and lines that constitute the boundary of the workspace. To do so, for a given curve (line or circle portion), belonging to a given B_i, a point lying on the curve is chosen, preferably not one of the end points. Then, it is verified that whether or not this point is located inside all the other B_is. This can be regarded as the most challenging part of the workspace determination that should be elaborated with care and is fully explained in [7].

Finally, applying the above procedure for different x'_H leads to obtaining the constant orientation workspace in three dimensional space. Fig. 10 represents the constant orientation

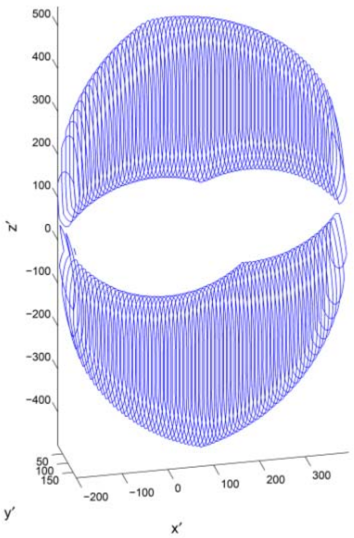

Figure 10. Constant orientation workspace for the design presented in TABLE I for φ =θ = 0.

workspace for a design presented in TABLE I for φ =θ = 0 and it can be seen that it is the same that is found with the CAD system in Fig. 6.

As the constant orientation workspace for a given cross section consists of some arcs and lines, the *Gauss Divergence Theorem* which can be applied to planar regions can be used to obtain the workspace area for all the sections located in the interval which is mentioned in (15). Finally the volume of the constant orientation workspace can be obtained using numerical integration of all the sections area. Fig. 11 represents the volume of the constant–orientation workspace with respect of two permitted orientations, (φ, θ), for the design presented in TABLE I.

VI. CONCLUSION

This paper investigated the FKP and constant–orientation workspace of 5-DOF parallel mechanisms (3T2R) with a limb kinematic arrangement of type RPUR. From the results for the IKP, two sets of simplified designs were presented whose FKP can be expressed either by a univariate expression or by a closed–form solution. Bohemian domes appeared in the geometrical interpretation of each limb and led to a CAD

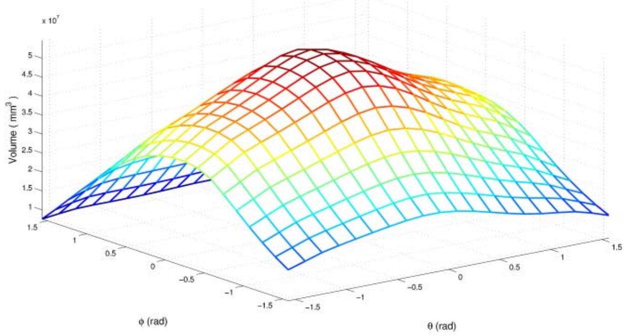

Figure 11. Volume of the constant orientation with respect of (φ, θ) for the design presented in TABLE I.

representation of the constant–orientation workspace. An algorithm was proposed, inspired from the one presented in [7], in order to find the boundary of the constant–orientation workspace which can be implemented in any computer algebra system. The algorithm made it possible to find the volume of the constant–orientation workspace by applying the *Gauss Divergence Theorem.* The principles of this paper can be applied equally well to the other types of symmetrical 5-DOF parallel mechanisms, such as 5-PRUR, in order to obtain similar results for the FKP. Ongoing works include the solution of the FKP in a univariate form for a general design and the study of the singular configurations.

REFERENCES

[1] O. Piccin, B. Bayle, B. Maurin, and M. de Mathelin, "Kinematic modeling of a 5-dof parallel mechanism for semi-spherical workspace," Mechanism and Machine Theory, 44(8):1485–1496, 2009.

[2] S. A. Joshi and L. W. Tsai, "Jacobian analysis of limited-dof parallel manipulators," ASME Journal of Mechanical Design, 124(2):254–258, 2002.

[3] Z. Huang and Q. C. Li, "General methodology for type synthesis of symmetrical lower-mobility parallel manipulators and several novel manipulators," The International Journal of Robotics Research, 21(2):131–145, 2002.

[4] Y. Fang and L. W. Tsai, "Structure synthesis of a class of 4-dof and 5-dof parallel manipulators with identical limb structures," The International Journal of Robotics Research, 21(9):799–810, 2002.

[5] Z. Huang and Q. C. Li, "Type synthesis of symmetrical lower mobility parallel mechanisms using the constraint–synthesis method," The International Journal of Robotics Research, 22(1):59–79, 2003.

[6] X. Kong and C. Gosselin, "Type synthesis of 5-dof parallel manipulators based on screw theory," Journal of Robotic Systems, 22(10):535–547, 2005.

[7] C. Gosselin, "Determination of the workspace of 6-dof parallel manipulators," ASME Journal of Mechanical Design, 112(3):331–336, 1990.

[8] A. Bonev and J. Ryu, "A geometrical method for computing the constant-orientation workspace of 6-prrs parallel manipulators," Mechanism and machine theory, 36(1):1–13, 2001.

[9] X. Kong and C. Gosselin, Type Synthesis of Parallel Mechanism, volume 33. Springer, Heidelberg, 2007.

[10] J.-P. Merlet, "Algebraic-geometry tools for the study of kinematics of parallel manipulators," In Computational Kinematics, pages 183–194. Kluwer Academic Publishers (J. Angeles c, G. Hommel and P. Kovacs Eds.), 1993.

[11] C. Gosselin, M. T. Masouleh, V. Duchaine, P. L. Richard, S. Foucault, and X. Kong, "Parallel mechanisms of the multipteron family: Kinematic architectures and benchmarking," In IEEE International Conference on Robotics and Automation, pages 555–560, Roma, Italy, 10-14 April 2007.

[12] M. T. Masouleh and C. Gosselin, "Kinematic analysis and singularity representation of 5-rprrr parallel mechanisms," In Fundamental Issues and Future Research Directions for Parallel Mechanisms and Manipulators, pages 79–90, Montpellier, France, 21–22 September 2008.

[13] J. P. Merlet, Parallel robots- open problems. In 9th Int. Symp. Of Robotics Research, pages 27–32, 9-12 Octobre 1999.

[14] Q. Jin, T. L. Yang, A. X. Liu, H. P. Shen, and F. H. Yao, "Structure synthesis of a class of 5–dof parallel robot mechanisms based on single opened–chain units," In Proceedings of the 2001 ASME Design Engineering Technical Conferences & Computers and Information in Engineering Conference, DETC2001/DAC21153, Pittsburgh, PA, 2001.