LAGRANGIAN DYNAMICS OF CABLE-DRIVEN PARALLEL MANIPULATORS: A VARIABLE MASS FORMULATION

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Abstract

In this paper, dynamic analysis of cable-driven parallel manipulators (CDPMs) is performed using the Lagrangian variable mass formulation. This formulation is used to treat the effect of a mass stream entering into the system caused by elongation of the cables. In this way, a complete dynamic model of the system is derived, while preserving the compact and tractable closed-form dynamics formulation. First, a general formulation for a CDPM is given, and the effect of change of mass in the cables is integrated into its dynamics. The significance of such a treatment is that a complete analysis of the dynamics of the system is achieved, including vibrations, stability, and any robust control synthesis of the manipulator. The formulation obtained is applied to a typical planar CDPM. Through numerical simulations, the validity and integrity of the formulations are verified, and the significance of the variable mass treatment in the analysis is examined. For this example, it is shown that the effect of introducing a mass stream into the system is not negligible. Moreover, it is non linear and strongly dependent on the geometric and inertial parameters of the robot, as well as the maneuvering trajectory.

Keywords: cable-driven robots; variable mass Lagrangian formulation; closed-form dynamics.

APPLICATION DE LA MÉTHODE DE LAGRANGE POUR LA MODÉLISATION DES ROBOTS À CABLES : UNE FORMULATION À MASSES VARIABLES

Résumé

Dans cet article, la modélisation dynamique des manipulateurs parallèles à câbles est présentée. L'effet de la variation de la longueur des câbles est pris en compte grâce à la méthode de Lagrange pour des systèmes à masses variables. Le modèle dynamique obtenu se présente alors sous une forme compacte. Cette modélisation est importante pour étudier en détail la dynamique du système aussi bien que pour des études de vibration, de stabilité et de conception de systèmes de commande robustes. Un exemple de modélisation d'un manipulateur à câbles plan est également présenté. Grâce à des simulations numériques, la validité et l'intégrité de la formulation obtenue sont d'abord vérifiés. Ensuite, l'effet de la variation de la masse est examiné. Pour ce faire, des simulations avec et sans l'effet de la variation de la masse sont considérées et les résultats sont comparés. Il est montré que pour l'exemple présenté, l'effet de la variation de la masse ne peut pas être négligé. Cette effet est non-linéaire et dépend fortement de la géométrie du manipulateur ainsi que de la trajectoire du robot.

Mots-clés : robots à câbles; formulation de Lagrange pour masses variables; dynamique.

1. INTRODUCTION

The equations of motion of constant mass systems can be derived using various classical approaches, such as the Newton-Euler, Lagrangian, virtual work, and Kane formulations. These basic principles of classical dynamics usually apply to systems comprising a definite number of objects with constant masses [1], and they can be extended to cases where the masses of the system components change. Such a complete treatment of the dynamic analysis of systems with variable mass is a challenging problem. The difficulties arise from the fact that the mass, centre of mass, and moments of inertia may vary in such mechanisms by a stream mass that is overtaken or expelled at a non-zero velocity, and that mass may change the linear and angular momentum of the overall system [2]. The dynamics of variable mass systems have been studied for a very long time. Some of their first applications were in applied mechanics, in systems of continuously variable mass, such as rockets [3], and most of the first works reported in this area are related to these applications. Meshchersky was among the first scientists to understand the modern dynamics of a rigid body with variable mass [4]. At the same time, in robotics applications, the motion of robots that pick up objects can be treated by varying mass dynamic analysis. Representative of this type of analysis is the work of McPhee in the dynamic analysis of multiple rigid bodies [5]. Djerassi [6] reported similar work in such applications. The most recent work reported in the area of variable mass systems has been performed by Cveticanin [2,4,7–10]. She studied the dynamics of body separation and developed an analytical procedure to determine the dynamic parameters of the remaining body after mass separation [10]. This method is based on the general principles of the momentum and angular momentum of a system of bodies. She also extended the Lagrangian formulation to systems of varying mass [2]. The latest reported work of Cveticanin and Djukic explains the extended kinematic and dynamic properties of a body in general motion [9], and presents their modification of the principle of linear and angular momentum conservation to obtain the linear and angular velocity of the body during mass separation. Furthermore, the dynamic analysis of cable-driven parallel manipulators (CDPMs) shows their inherent complexity due to their closed-loop structure and kinematic constraints. Although the dynamic analysis of such manipulators is essential for stability analysis and closed loop control synthesis, little work has been reported on the dynamic analysis of CDPMs [11–14]. In these manipulators, a change in cable length causes the effective mass of their limbs to continuously vary in time. Moreover, the varying mass of the cables is a function of the position of the moving platform. In all the work reported in the dynamics of CDPMs, the effect of varying mass in cables has been neglected, because of the small changes of mass in the cables. However, in some applications, such as the large adaptive reflectors used in the next generation of giant telescopes [12], the cables can be as long as 1,000 meters, and so the mass variation of cables could play an important role for this class of systems. To evaluate the aforementioned importance, a model that takes into account the mass variation is necessary. However, the variation of mass is not the only effect that may affect this particular class of systems. In fact, the long cables can also introduce considerable sag effect [13]. To evaluate these effects accurately, they should be considered one at a time.

In this paper, the dynamic equations of CDPMs are discussed in detail, taking into account the mass variation of the cables, but neglecting the sag effect. This simplification assumption is not necessarily valid, but at this stage, we focus on the mass variation effect only. According to this assumption, the dynamics is expressed in terms of the Lagrangian formulation, and a set of compact and closed-form formulations is obtained. This general formulation is then adopted for modeling a typical planar CDPM, for which a simulation study is performed. It is shown that the effect of a mass stream entering into the system is not negligible: it is nonlinear and strongly dependent on the geometric and inertial parameters of the robot, and on the maneuvering trajectory.

2. KINEMATIC ANALYSIS OF CDPMS

The general structure of CDPMs that is used in this paper is shown in Fig. 1. In this manipulator, the moving platform is supported by n limbs (cables) of identical kinematic structure, while the limbs are considered as rigid slender rods for the sake of dynamic analysis. The kinematic structure of the limb may be considered as spherical-prismatic-spherical (commonly denoted as SPS), in which only the prismatic joint is actuated. The kinematic structure of a prismatic joint is used to model the elongation of each link. As shown in Fig. 2, A_i denote the fixed base points of the cables, B_i denote the points of attachment of the cables to the moving platform, and $\mathbf{l} = [l_1 \cdots l_n]^T$ denotes the vector of the cable lengths. Moreover, the position vector of the moving platform frame $\{p\}$, as well as the cable frame $\{c_i\}$, are defined as $\begin{bmatrix} \mathbf{x}_p^T & \mathbf{x}_c^T \end{bmatrix}^T$, in which $\mathbf{x}_p$ denotes the position of the moving platform according to the base frame $\{0\}$, and $\mathbf{x}_c = \begin{bmatrix} \mathbf{x}_{c1}^T & \cdots & \mathbf{x}_{cn}^T \end{bmatrix}^T$ denotes the vector of the cable coordinates where $\mathbf{x}_{c1}$ is the position of the cable's centre c_i, according to the base frame (see Fig. 2). Similarly, the angular coordinates of the moving platform $\{p\}$ and the cables $\{c_i\}$ relative to the base frame are defined as $\begin{bmatrix} \boldsymbol{\varphi}_p^T & \boldsymbol{\varphi}_c^T \end{bmatrix}^T$, in which $\boldsymbol{\varphi}_p = [\gamma, \beta, \alpha]^T$ are any user-defined Euler angles of the moving platform, and $\boldsymbol{\varphi}_c = \begin{bmatrix} \boldsymbol{\varphi}_1^T & \cdots & \boldsymbol{\varphi}_n^T \end{bmatrix}^T$ is the angle vector of the cable frame $\{c_i\}$ which is attached to the center of the cables. Subsequently, each angle vector $\boldsymbol{\varphi}_i$, is defined by its three Euler angles: $\boldsymbol{\varphi}_i = [\gamma_i, \beta_i, \alpha_i]^T$. Accordingly, the rotational matrix of moving platform coordinate $\{p\}$, ${}^o\mathbf{R}_p$ and cables coordinate $\{c_i\}$, ${}^o\mathbf{R}_{ci}$ relative to the base frame defined as:

$$\mathbf{R}(\gamma, \beta, \alpha) = {}^o\mathbf{R}_p, \; \mathbf{R}(\gamma_i, \beta_i, \alpha_i) = {}^o\mathbf{R}_{ci}. \tag{1}$$

As explained in [12,15], and [16], the inverse kinematics of CDPMs, like that of any other parallel manipulator, can be obtained by writing the loop closure equations. Therefore, according to the Fig. 2 the vector loop closure formulation can be written as:

$$\mathbf{x}_p + \mathbf{E}_i = \mathbf{x}_{Ai} + l_i\mathbf{S}_i, \; \mathbf{E}_i = {}^o\mathbf{R}_p {}^p\mathbf{x}_{Bi}, \tag{2}$$

where $l_i\mathbf{S}_i = A_iB_i$, $\mathbf{x}_{Ai}$ and ${}^p\mathbf{x}_{Bi}$ are respectively the position vectors of A_i relative to the $\{0\}$ and the position vector of B_i relative to $\{p\}$. Since it is assumed that the z axis of the $\{c_i\}$ frame is

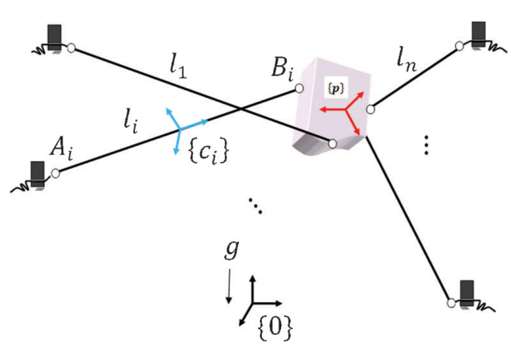

Fig. 1. General structure of cable-driven parallel manipulators (CDPMs).

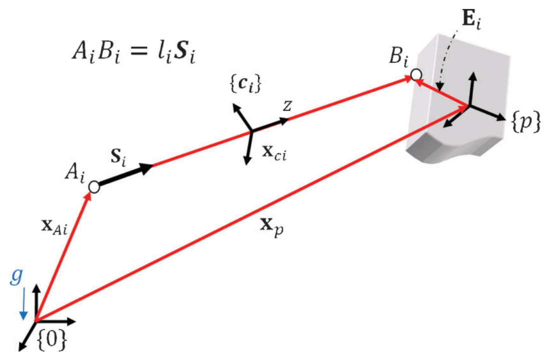

Fig. 2. A single limb in a cable-driven parallel manipulator.

directed along the cable length, the geometric model can be completed by the two following equations:

$$\mathbf{x}_{ci} = \mathbf{x}_{Ai} + \frac{l_i \mathbf{S}_i}{2}, \ {}^{o}\mathbf{R}_{ci}[0 \quad 0 \quad l_i]^T = l_i \mathbf{S}_i. \tag{3}$$

Equatons (1–3) allow all the coordinates of the system to be expressed as a function of the generalized coordinates. By choosing $\mathbf{x} = \begin{bmatrix} \mathbf{x}_p^T & \boldsymbol{\varphi}_p^T \end{bmatrix}^T \in \mathbb{R}^m$ (position and orientation of the moving platform) as generalized coordinates, we obtain:

$$\mathbf{x}_c = \mathbf{f}_x(\mathbf{x}), \ \boldsymbol{\varphi}_c = \mathbf{f}_\varphi(\mathbf{x}), \ \mathbf{l} = \mathbf{f}_l(\mathbf{x}), \tag{4}$$

where $\mathbf{f}_x$, $\mathbf{f}_\varphi$ and $\mathbf{f}_l$ are kinematic equations obtained from the loop closure. The time derivative of Eqs. (1–4), combined with the cross product propriety of rotation matrix (i.e. ${}^{o}\mathbf{R}_a{}^{T\,o}\dot{\mathbf{R}}_a = \boldsymbol{\omega}_a \times \mathbf{y}$ where $\times$ denotes the cross product operator and $\boldsymbol{\omega}_a$ is the angular velocity of frame {a} expressed in terms of frame {a}) may lead to a relation that expresses the linear and angular velocities of the frame attached to the cables center of mass, as well as the time derivative of the cable lengths, as function of the linear and angular velocities of the moving platform:

$$\begin{bmatrix} \dot{\mathbf{x}}_c \\ {}^{c}\boldsymbol{\omega}_c \end{bmatrix} = \begin{bmatrix} \mathbf{J}_{xx}(\mathbf{x}) & \mathbf{J}_{x\omega}(\mathbf{x}) \\ \mathbf{J}_{\omega x}(\mathbf{x}) & \mathbf{J}_{\omega\omega}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_p \\ {}^{p}\boldsymbol{\omega}_p \end{bmatrix}, \ \dot{\mathbf{l}} = [\,\mathbf{J}_{lx}(\mathbf{x}) \quad \mathbf{J}_{l\omega}(\mathbf{x})\,] \begin{bmatrix} \dot{\mathbf{x}}_p \\ {}^{p}\boldsymbol{\omega}_p \end{bmatrix}, \tag{5}$$

where $\mathbf{J}_{xx}$, $\mathbf{J}_{x\omega}$, $\mathbf{J}_{\omega x}$, $\mathbf{J}_{\omega\omega}$, $\mathbf{J}_{lx}$ and $\mathbf{J}_{l\omega}$ are Jacobian matrices; $\dot{\mathbf{x}}_c$ and $\dot{\mathbf{x}}_p$ are the linear velocities of the cables and the moving platform respectively, and ${}^{c}\boldsymbol{\omega}_c$ and ${}^{p}\boldsymbol{\omega}_p$ are the angular velocities expressed in the cables and moving platform frame respectively. In order to eliminate the velocities of the cable in the Lagrangian formulation presented below, Eq. (5) is used to collect all the linear velocities of the cables and the moving platform as function of only the linear and angular velocities of the moving platform:

$$\begin{bmatrix} \dot{\mathbf{x}}_p \\ \dot{\mathbf{x}}_c \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{J}_{xx}(\mathbf{x}) & \mathbf{J}_{x\omega}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_p \\ {}^{p}\boldsymbol{\omega}_p \end{bmatrix}, \tag{6}$$

where **1** represents the identity matrix. Similarly, the angular velocities of the cables and the moving platform are rewritten as:

$$\begin{bmatrix} {}^{p}\boldsymbol{\omega}_p \\ {}^{c}\boldsymbol{\omega}_c \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{J}_{\omega x}(\mathbf{x}) & \mathbf{J}_{\omega\omega}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_p \\ {}^{p}\boldsymbol{\omega}_p \end{bmatrix}. \tag{7}$$

Conveniently for the Lagrangian formulation, Eqs. (4) and (5) can be expressed as a function of the derivative of the generalized coordinates. In order to achieve this, the following relation between the derivative of the Euler angles and the angular velocity can be established [15]:

$${}^{p}\boldsymbol{\omega}_p = \mathbf{J}_{\omega\varphi}(\mathbf{x})\dot{\boldsymbol{\varphi}}_p. \tag{8}$$

This equation can then be used to rewrite (4) and (5) as:

$$\begin{bmatrix} \dot{\mathbf{x}}_p \\ \dot{\mathbf{x}}_c \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{J}_{xx}(\mathbf{x}) & \mathbf{J}_{x\omega}(\mathbf{x})\mathbf{J}_{\omega\varphi}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_p \\ \dot{\boldsymbol{\varphi}}_p \end{bmatrix} = \mathbf{J}_x(\mathbf{x})\dot{\mathbf{x}}, \tag{9}$$

$$\begin{bmatrix} {}^{p}\boldsymbol{\omega}_p \\ {}^{c}\boldsymbol{\omega}_c \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{J}_{\omega\varphi}(\mathbf{x}) \\ \mathbf{J}_{\omega x}(\mathbf{x}) & \mathbf{J}_{\omega\omega}(\mathbf{x})\mathbf{J}_{\omega\varphi}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_p \\ \dot{\boldsymbol{\varphi}}_p \end{bmatrix} = \mathbf{J}_{\varphi}(\mathbf{x})\dot{\mathbf{x}}. \tag{10}$$

3. KINETIC ENERGY OF CDPMS

In order to derive the kinetic energy of the system, the kinetic energy of the robot components are derived and added. A CDPM consists of a moving platform and several limbs, in which the limbs are modeled as rigid slender rods. The actuators, which are often composed of motors combined with pulleys, are not included in our formulation. Therefore, the mass of all the objects in the mechanism can be expressed as:

$$\mathbf{M}(\mathbf{l}) = \begin{bmatrix} \mathbf{M}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_c(\mathbf{l}) \end{bmatrix}, \tag{11}$$

in which $\mathbf{M}_p$ and $\mathbf{M}_c$ denote the mass matrices of the moving platform and all the cables respectively:

$$\mathbf{M}_p = \begin{bmatrix} m_p & 0 & 0 \\ 0 & m_p & 0 \\ 0 & 0 & m_p \end{bmatrix}, \mathbf{M}_c(\mathbf{l}) = \begin{bmatrix} m_{c1}(l_1)\mathbf{1}_3 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & m_{cn}(l_n)\mathbf{1}_3 \end{bmatrix}. \tag{12}$$

In this definition, m_p is the moving platform mass and m_{c1} is the mass of the cables expressed as a function of its linear density ρ_m and its lengths l_i, as follows:

$$m_i(l_i) = \rho_m l_i. \tag{13}$$

Similarly, the moment of inertia of all the components of a CDPM can be collected into:

$$\mathbf{I}(\mathbf{l}) = \begin{bmatrix} \mathbf{I}_p & 0 \\ 0 & \mathbf{I}_c(\mathbf{l}) \end{bmatrix}, \tag{14}$$

where $\mathbf{I}_p$ and $\mathbf{I}_c$ are the inertial matrices of the moving platform and the cables respectively, given by:

$$\mathbf{I}_p = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}, \mathbf{I}_c(\mathbf{l}) = \begin{bmatrix} \mathbf{I}_{c1}(l_1) & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \mathbf{I}_{cn}(l_n) \end{bmatrix}. \tag{15}$$

Since the cables are modeled as slender rods and with the assumption that the z axis of the $\{c_i\}$ frame is directed along the cable length, the moment of inertia of the cables $\mathbf{I}_{ci}$ is defined as:

$$\mathbf{I}_{ci}(l_i) = \frac{\rho_m}{12} \begin{bmatrix} l_i^3 & 0 & 0 \\ 0 & l_i^3 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{16}$$

According to Eq. (2), l_i can be expressed as a function of the generalized coordinates. Thus, the total kinetic energy for all the components of a CDPM can be expressed as:

$$T = \frac{1}{2}\left[\begin{bmatrix} \dot{\mathbf{x}}_p \\ \dot{\mathbf{x}}_c \end{bmatrix}^T \mathbf{M}(\mathbf{x}) \begin{bmatrix} \dot{\mathbf{x}}_p \\ \dot{\mathbf{x}}_c \end{bmatrix} + \begin{bmatrix} {}^p\boldsymbol{\omega}_p \\ {}^c\boldsymbol{\omega}_c \end{bmatrix}^T \mathbf{I}(\mathbf{x}) \begin{bmatrix} {}^p\boldsymbol{\omega}_p \\ {}^c\boldsymbol{\omega}_c \end{bmatrix} \right]. \tag{17}$$

The substitution of the Jacobian matrices defined by Eqs. (7) and (8) leads to:

$$T = \frac{1}{2}\dot{\mathbf{x}}^T \mathbf{D}(\mathbf{x})\dot{\mathbf{x}}, \tag{18}$$

where the mass matrix of the system is given by:

$$\mathbf{D}(\mathbf{x}) = \mathbf{J}_x{}^T(\mathbf{x})\mathbf{M}(\mathbf{x})\mathbf{J}_x(\mathbf{x}) + \mathbf{J}_\varphi{}^T(\mathbf{x})\mathbf{I}(\mathbf{x})\mathbf{J}_\varphi(\mathbf{x}). \tag{19}$$

4. VARIABLE MASS LAGRANGIAN APPROACH

In this section, the dynamics of a cable-driven parallel manipulator is obtained by the variable mass Lagrangian formulation. As the length of the cables in a CDPM is a function of the position of the moving platform, the cable mass changes in time. In fact, the mass that is added to or removed from the system will add momentum to the system or remove momentum from it. The dynamics of the mechanism with variable mass is discussed in detail in [2] by Cveticanin, who extends the Lagrangian formulation to:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{x}}}\right)-\left(\frac{\partial T}{\partial \mathbf{x}}\right)=\mathbf{q}+\mathbf{q}^{Fi}+\mathbf{d}+\mathbf{q}^{R^*}. \tag{20}$$

In this formulation, $\mathbf{q}$ and $\mathbf{q}^{Fi}$ are the generalized forces caused by non conservative and non-conservative external forces acting on the system respectively. Furthermore, $\mathbf{d}+\mathbf{q}^{R^*}$ accounts for the effect of changing mass in the system. In other words, $\mathbf{q}^{R^*}$ is an impact force that is caused by the mass stream entering into the system or being expelled from it, and is a function of the mass variation and its relative velocity. Furthermore, $\mathbf{d}$ accounts for the direct energy that is added to or removed from the system by entry or departure of the stream mass.

4.1. Kinetic Energy Terms

Let us examine the required terms of the Lagrangian formulation for a CDPM. As usual, the first two terms can be derived from the kinetic energy of the system given by (16):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\mathbf{x}}}\right)-\left(\frac{\partial T}{\partial \mathbf{x}}\right)=\mathbf{D}(\mathbf{x})\ddot{\mathbf{x}}+\left(\dot{\mathbf{D}}(\mathbf{x})-\frac{1}{2}\frac{\partial}{\partial \mathbf{x}}\left(\dot{\mathbf{x}}^T\mathbf{D}(\mathbf{x})\right)\right)\dot{\mathbf{x}}, \tag{21}$$

where $\dot{\mathbf{D}}(\mathbf{x})$are the time derivatives of the terms given by (17).

4.2. Generalized Forces

As explained for the extended Lagrangian formula, $\mathbf{q}^{Fi}$ and $\mathbf{q}$ are the generalized forces caused by conservative and non-conservative external forces acting on the system respectively. The generalized force acting on the system caused by external non conservative forces is composed of the elements $\mathbf{w}_x+\mathbf{q}_{nc}$, such that:

$$\mathbf{q}=\mathbf{w}_x+\mathbf{q}_{nc}, \tag{22}$$

where $\mathbf{w}_x$ is the wrench (forces and torques) corresponding to the projection of the cable force on the platform, and $\mathbf{q}_{nc}$ represents the external forces and torques acting directly on the moving platform. According to the principle of virtual work and the Jacobians given by Eqs. (3) and (6), the vector $\mathbf{w}_x$ can be obtained by projecting the cable forces into the Cartesian space, using the manipulator Jacobian matrices as follows:

$$\mathbf{w}_x=[\,\mathbf{J}_{lx}(\mathbf{x})\quad \mathbf{J}_{l\omega}(\mathbf{x})\mathbf{J}_{\omega\varphi}(\mathbf{x})\,]^T\boldsymbol{\tau}=\mathbf{J}_w{}^T(\mathbf{x})\boldsymbol{\tau}, \tag{23}$$

where $\boldsymbol{\tau}$ denotes the vector of the cable forces. The contribution of the gravitational forces may be expressed as the following equation of potential energy:

$$V=\mathbf{g}^T\left(\mathbf{M}_p\mathbf{x}_p+\sum_{i=1}^{n}m_i(l_i)\mathbf{x}_{ci}\right), \tag{24}$$

where **g** is the gravity vector represented in the base frame. According to [2], potential energy can be expressed as a function of the generalized coordinates. Therefore, $\mathbf{q}^{Fi}$ is obtained by the partial derivative of the potential energy with respect to the generalized coordinates:

$$\mathbf{G}(\mathbf{x}) = -\frac{dV}{d\mathbf{x}}. \tag{25}$$

4.3. Variable Mass Terms

The formulation proposed for the varying mass mechanism in [2] was defined for a particle mass system. The additional terms required to accommodate the variable mass mechanism are only a function of mass derivatives (a small variation in mass divided by a small variation in time). For this reason, and because these variations are continuous, the mass derivative acts as a particle, even for a body system. This interpretation has already been considered in [2,17] for the analysis of the vibration of varying mass mechanisms (see also [4]). As discussed in [2], the effect of changing mass in the system is caused by a variable momentum. This effect can be divided into the impact forces denoted by $\mathbf{q}^{R^*}$and the energy that was added or removed from the system by the variable mass, denoted by **d**. Since cables are the only source of variable mass and the variation is only function of the generalized coordinates, $\boldsymbol{d_k}$ can be determined by [17]:

$$d_k(\mathbf{x},\dot{\mathbf{x}}) = -\frac{1}{2}\sum_{i=1}^{n}\frac{\partial m_i(l_i)}{\partial x_k}\mathbf{v}_i{}^T\mathbf{v}_i, \tag{26}$$

where $\mathbf{v}_i$ is the velocity of the variable mass i and k denote individual generalized coordinates. According to Fig. 1, this mass variation is located at the beginning of the cable $i$and its velocity is in only one direction when it is expressed in the frame of the cable. For this reason, $\mathbf{v}_i$ can be considered as a scalar given by $\dot{l}_i$. Then, using Eqs. (2) and (11), Eqs. (24) can be rewritten as:

$$d_k(\mathbf{x},\dot{\mathbf{x}}) = -\frac{1}{2}\rho_m\sum_{i=1}^{m}\frac{\partial f_{li}}{\partial x_k}\left(\frac{\partial f_{li}}{\partial \mathbf{x}}\dot{\mathbf{x}}\right)^2. \tag{27}$$

Now, the effect of the impact forces $q_k^{R^*}$ can be obtained from [17]:

$$q_k^{R^*}(\mathbf{x},\dot{\mathbf{x}}) = \sum_{i=1}^{m} m_i(l_i)\mathbf{v}_{oi}{}^T\frac{\partial \mathbf{p}_i}{\partial x_k}, \tag{28}$$

where $\mathbf{v}_{oi}$ is the velocity of the expelled or gained mass, and **p** is the position of the mass variation. This variation is also located at the beginning of the cable i and its position variation, and its velocity is in only one direction when they are expressed in the frame of the cable. For this reason, $\mathbf{v}_{oi}$ and the variation of p_i can be interpreted as scalars, given by $\boldsymbol{i}_i$ and $\partial l_i/\partial x_k$ respectively. Then, using Eqs. (2) and (11), Eq. (26) can be rewritten as:

$$q_k^{R^*}(\mathbf{x},\dot{\mathbf{x}}) = \rho_m \sum_{i=1}^{m} \left(\frac{\partial f_{li}}{\partial \mathbf{x}}\dot{\mathbf{x}}\right)^2 \frac{\partial f_{li}}{\partial x_k}. \tag{29}$$

d_k and $q_k^{R^*}$ can be combined, as follows:

$$d_k(\mathbf{x},\dot{\mathbf{x}}) + q_k^{R^*}(\mathbf{x},\dot{\mathbf{x}}) = \frac{1}{2}\rho_m \sum_{i=1}^{m} \frac{\partial f_{li}}{\partial x_k}\left(\frac{\partial f_{li}}{\partial \mathbf{x}}\dot{\mathbf{x}}\right)^2. \tag{30}$$

4.4. Final Dynamics Equations

From Eqs. (17–19,21,23), and (28), the general form of the dynamics of CDPM can be released in compact standard form, as:

$$\mathbf{D}(\mathbf{x})\ddot{\mathbf{x}} + \mathbf{c}(\mathbf{x},\dot{\mathbf{x}}) + \mathbf{G}(\mathbf{x}) = \mathbf{J}_w(\mathbf{x})\boldsymbol{\tau} + \mathbf{q}_{nc}, \tag{31}$$

where $\mathbf{D}$ is given by Eq. (17), $\mathbf{G}$ is given by Eq. (23), $\mathbf{J}_w$ is defined by Eq. (21), and $\mathbf{c}$ is given by:

$$\mathbf{c}(\mathbf{x},\dot{\mathbf{x}}) = \left(\dot{\mathbf{D}}(\mathbf{x}) - \frac{1}{2}\frac{\partial}{\partial \mathbf{x}}\left(\dot{\mathbf{x}}^T\mathbf{D}(\mathbf{x})\right)\right)\dot{\mathbf{x}}\text{-}\mathbf{d}(\mathbf{x},\dot{\mathbf{x}}) + \mathbf{q}^{R^*}(\mathbf{x},\dot{\mathbf{x}}), \tag{32}$$

where each element of $\mathbf{d}+\mathbf{q}^{R^*}$ is given by Eq. (28). In Eq. (30), $\mathbf{D}$ is the mass matrix; $\mathbf{c}$ is the vector of the centrifugal, Coriolis, and mass variation terms; and $\mathbf{G}$ is the vector of the gravity terms. Finally, $\mathbf{q}_{nc}$ is the external wrench vector acting directly on the moving platform.

5. CASE STUDY

In this section, the dynamics of the planar CDPM discussed in [12] (see Fig. 3) was considered. This CDPM is a simplified planar version adopted from the structure of the Large Adaptive Reflector (LAR). This structure consists of parallel redundant manipulators

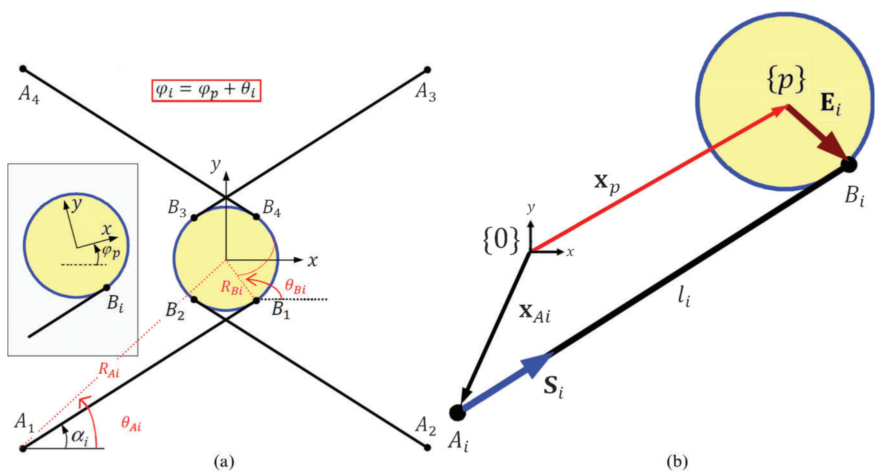

Fig. 3. (a) Simple schematic representation of the planar CDPM; (b) Vectors definitions for Jacobian derivation of the manipulator.

actuated by long cables. The control objective in the simplified mechanism is to track the position and orientation of the moving platform, as desired, in the presence of disturbance forces, such as wind turbulence. The geometric and inertial parameters used in the simulations of the system have been adopted from the LAR design. In this way, the length of the cables is in the order of one kilometer. The main control purpose is the positioning of the moving platform, $\mathbf{x}=[x \quad y \quad \varphi]^T$. The geometric and inertial parameters of the system are explained in Table 1. In this subsection at first, the dynamics of the planar CDPM is obtained by the Lagrangian method. Then, the effect of the variable mass in the cables is studied in detail.

From the inverse kinematic analysis, the length of the cable l_i and the angle α_i can be obtained easily by writing the loop closure equations as follows:

$$\begin{aligned} l_i &= \left[(x+R_B\cos\varphi_i-x_{Ai})^2+(y+R_B\sin\varphi_i-y_{Ai})^2\right], \\ \alpha_i &= \mathrm{atan2}((y+R_B\sin\varphi_i-y_{Ai}),(x+R_B\cos\varphi_i-x_{Ai})), \end{aligned} \tag{33}$$

where $\mathbf{x}_{Ai}=[x_{Ai} \quad y_{Ai}]^T$ is the position of fixed points A_i.

In order to obtain the Jacobian matrix, let us differentiate the vector loop Eq. (2) with respect to time:

$$[\dot{x} \quad \dot{y}]^T+\dot{\varphi}_p(\mathbf{K}\times\mathbf{E}_i)=\dot{l}_i\mathbf{S}_i+\dot{\alpha}_i l_i(\mathbf{K}\times\mathbf{S}_i), \tag{34}$$

where, $\mathbf{K}$ is the unit vector in z direction. vectors $\mathbf{E}_i$ and $\mathbf{S}_i$ are defined as follows:

$$\begin{aligned} \mathbf{E}_i &= [E_{ix} \quad E_{iy}]^T=[R_B\cos(\varphi_i) \quad R_B\sin(\varphi_i)]^T, \\ \mathbf{S}_i &= [S_{ix} \quad S_{iy}]^T=[\cos(\alpha_i) \quad \sin(\alpha_i)]^T, \end{aligned} \tag{35}$$

In order to obtain expressions for $\dot{l}_i$ and $\dot{\alpha}_i$, dot multiply and cross multiply both sides of equation by $\mathbf{S}_i$ respectively, we have:

Table 1. Geometric and inertial parameters of the system.

	Symbols	Quantity
Radius of location circle of fixed points A_i's	R_A	900 m
Radius of location circle of moving platform points B_i's	R_B	10 m
Angle of fixed points A_i	θ_A	[−135°,−45°,45°,135°]
Angle of moving platform points B_i	θ_B	[−45°,−135°,135°,45°]
The moving platform mass	M_p	2500 kg
The moving platform moments	I_p	3.5e105 kg/m
Mass density of the cables	ρ_m	0.215 kg/m

$$\mathbf{J}_{xx}=\left[\begin{matrix} S_{ix} & S_{iy}\end{matrix}\right]_{i=1}^{4},\ \mathbf{J}_{\omega x}=\left[E_{ix}S_{iy}-E_{iy}S_{ix}\right]_{i=1}^{4},$$
$$\mathbf{J}_{\omega x}=\left[\begin{matrix} -\frac{1}{l_i}S_{iy} & -\frac{1}{l_i}S_{ix}\end{matrix}\right]_{i=1}^{4},\ \mathbf{J}_{\omega\omega}=\left[-\frac{1}{l_i}\left(E_{ix}S_{iy}+E_{iy}S_{ix}\right)\right]_{i=1}^{4}. \tag{36}$$

Moreover, for planar CDPMs, we have $\mathbf{J}_{\omega\varphi}=\mathbf{I}$, and the Jacobian matrices are therefore easily defined by Eqs. (7) and (8). Finally, by deriving Eqs. (17,30), and (23), the mass matrix $\mathbf{D}$, the centrifugal, Coriolis, and mass variation terms $\mathbf{c}$, and the gravity vector terms $\mathbf{G}$ are obtained. Thus, the dynamic modeling of planar CDPM is expressed as follows:

$$\mathbf{D}(\mathbf{x})_{3\times 3}\ddot{\mathbf{x}}_{3\times 1}+\mathbf{c}(\mathbf{x},\dot{\mathbf{x}})_{3\times 1}+\mathbf{G}(\mathbf{x})=\left[\begin{matrix} F_x & F_y & \tau_z\end{matrix}\right]^T+\mathbf{q}_{nc}, \tag{37}$$

where F_x, F_y, and τ_z form the wrench applied on the moving platform, defined by:

$$\left[\begin{matrix} F_x & F_y & \tau_z\end{matrix}\right]^T=\mathbf{J}_W(\mathbf{x})_{3\times 4}{}^T\boldsymbol{\tau}_{4\times 1}. \tag{38}$$

In Eq. (35), $\boldsymbol{\tau}_{4\times 1}$ is the vector of the forces in joint space or, in other words, the tensions in the cables that are generated by the actuators (motors). As the Jacobian matrix in a redundant manipulator is non-square, tension in the cables can be obtained by the algorithms of Redundancy Resolution (optimal distribution of forces in cables) [19].

This resolution also ensures positive tension in all cables. In fact the redundancy resolution is formulated as an optimization problem with equality and non-equality constraints caused by CDPM structure and cables restriction. The equality constraint is the Jacobian transformation between cable forces, $\boldsymbol{\tau}_{4\times 1}$ and the force acting on the moving platform $\left[\begin{matrix} F_x & F_y & \tau_z\end{matrix}\right]^T$. The non-equality constraint ensures the positive cables forces. In other words, the redundancy resolution problem has been considered as the problem of computing the minimum norm cables forces with the following equality and non-equality constraints:

$$\min \|\boldsymbol{\tau}_{4\times 1}\|_2$$

under the constraints:

$$\left[\begin{matrix} F_x & F_y & \tau_z\end{matrix}\right]^T=\mathrm{J}_W(\mathrm{x})_{3\times 4}^T\tau_{4\times 1},\tau_{4\times 1}>0. \tag{39}$$

The numerical solution of this optimization problem is explained in detail, as example, in [18].

For simulation, a specific displacement of the moving platform is chosen. This simple trajectory is shown in Fig. 4. Then, the forces in Cartesian space are obtained by the inverse dynamic model given by Eq. (34). These forces are compared with the forces obtained by the same simulation, in which the effect of variable masses in the cables is neglected. Figure 5(a) shows the forces and torque in Cartesian space. Figure 5(b) shows the projected forces in the links space. In other words, it shows the tensions in the cables that are defined by Eq. (35) as $\boldsymbol{\tau}=\left[\begin{matrix}\tau_1 & \tau_2 & \tau_3 & \tau_4\end{matrix}\right]^T$. These forces were obtained by driving the numerical algorithm used to solve the "non negative least-squares constraints problem" described in [19] and implemented in the Matlab optimization toolbox. As we expect from the dynamics equation analysis, the

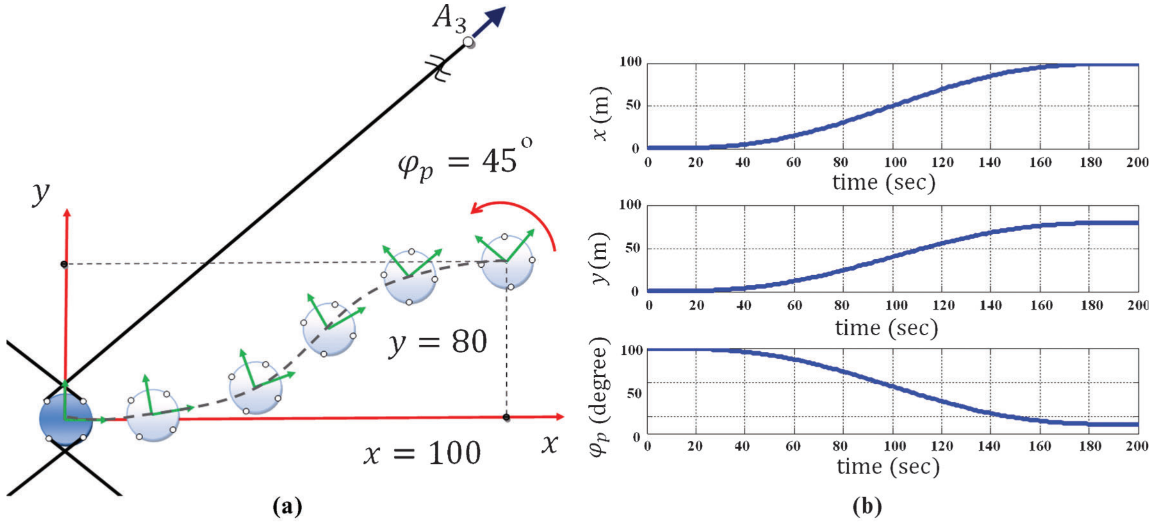

Fig. 4. Desired trajectory.

variable mass has a significant effect on the dynamics of the manipulator. In applications such as the LAR project [12], the length and mass density of the cables are important. In this context, the variable mass of the cables plays a vital role in the dynamics of the CDPM. Moreover, the effects of the variable mass in the cables are strongly dependent on the position and velocity trajectories. This effect is non linear, and dependent on parameters like the mass density of the cables, the mass of the moving platform, and the kinematic structure. In fact, the additional effect of the variable mass is completely described by Eq. (28). Therefore, this effect is directly proportional to the cable mass density ρ_m. This parameter could reduce the effect of the variable mass. However, such a reduction would increase the flexibility of the cables, which is not necessarily a better outcome. In addition, since $\mathbf{f}_l(\mathbf{x})$ in Eq. (2) is a kinematic function of the position of the moving platform, the variable mass effect is strongly dependent on the size and topology of the CDPM.

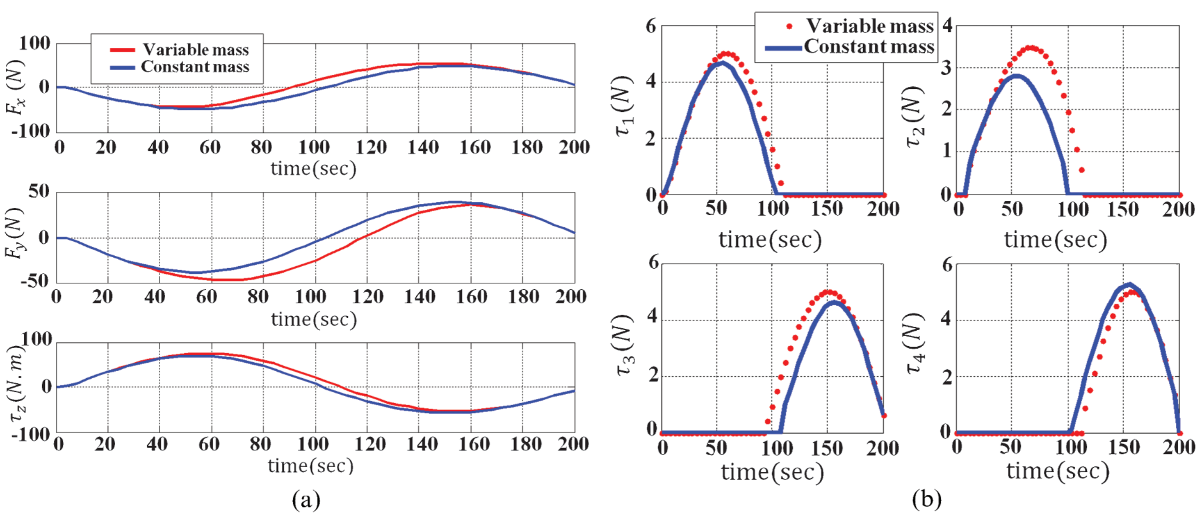

Fig. 5. (a) Forces and torque in Cartesian space (moving platform workspace); (b) tension in the cables (forces in the joint space).

6. CONCLUSIONS

This paper focused on the dynamic modeling of cable-driven parallel manipulators (CDPMs) using the Lagrangian formulation. In previous works, the effect of a mass stream entering into the system caused by elongation of the cables is neglected, whereas in this paper, this effect is treated using a Lagrangian variable mass formulation. In this way, a complete dynamics of the system is derived, while the compact and tractable closed form dynamics formulation is preserved. First, a general formulation for a general CDPM is given, where the effect of mass variation of the cables is integrated into its dynamics. The significance of such a treatment can be appreciated in a complete analysis of the dynamics, vibrations, and stability of such systems, and in any robust control synthesis of these manipulators. The general formulation is applied to a typical planar CDPM with cables 900 meters in length. Through simulation, the validity and integrity of the formulation obtained are verified, and the significance of variable mass treatment in such an analysis is examined. It is shown that the effect of a mass stream entering into the system is not negligible: it is non-linear and strongly dependent on the geometric and mass parameters of the robot, and on the maneuvering trajectory. However, the long cables do introduce additional sag effect, which is neglected in this paper. In future work, this effect will be studied in detail by using finite element approach.

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