CONTROLLABLE WORKSPACE OF CABLE-DRIVEN REDUNDANT PARALLEL MANIPULATORS BY FUNDAMENTAL WRENCH ANALYSIS

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ABSTRACT

Workspace analysis is always a crucial issue in robotic manipulator design. This paper introduces a set of newly defined fundamental wrenches that opens new horizons to physical interpretation of controllable workspace of a general cable-driven redundant parallel manipulator. Based on this set of fundamental wrenches, a novel tool is presented to determine configurations of cable-driven redundant parallel manipulator that belong to the controllable workspace. Analytical expressions of such workspace boundaries are obtained in an implicit form and a rigorous mathematical proof is provided for this method. Finally, the proposed method is implemented on a spatial cable-driven manipulator of interest.

Keywords: fundamental wrench; controllable workspace; cable-driven parallel manipulator.

DÉTERMINATION DE L'ESPACE DE TRAVAIL COMMANDABLE D'UN ROBOT PARALLÈLE À CÂBLES REDONDANT PAR L'ANALYSE DE TORSEURS D'EFFORT

RÉSUMÉ

L'analyse de l'espace de travail est toujours d'une importance cruciale lors de la conception d'un robot manipulateur. Cet article présente un nouvel ensemble de torseurs d'efforts permettant une nouvelle interprétation physique de l'espace de travail commandable d'un robot parallèle à câbles redondant général. Se basant sur cet ensemble de torseurs d'efforts, un nouvel outil est présenté qui détermine les configurations d'un robot parallèle à câbles redondant comprises dans l'espace de travail commandable. Les expressions analytiques des frontières de cet espace de travail sont obtenues sous forme implicite et une preuve mathématique rigoureuse de cette méthode est démontrée. Enfin, la méthode est implémentée sur un robot à câbles spatial.

Mots-clés : torseur d'effort ; espace de travail commandable ; robot parallèle à câbles.

1. INTRODUCTION

Cable-driven parallel manipulators (CDRPMs) are a special class of parallel robots in which rigid extensible links are replaced by actuated cables. This type of parallel manipulators has several attractive features and some advantage such as large workspace, high payload-to-weight ratios, and low inertial properties, compared to those of conventional parallel manipulators. Having these desirable characteristics, cable robots have potential for many real world applications such as heavy payload handling [1], manufacturing operations [2], automated construction systems [3], high-speed manipulation [4,5], and locomotion interfaces [6]. In cable parallel manipulators, external loading can be used to keep the cable in tension, in which the loading can be obtained from a redundant cable [7] or from another force-applying element such as a spring [8], a pneumatic cylinder [9], or a helium aerostat [10].

Besides many advantages and promising potential, there are some new challenges in designing and development of cable manipulators. Workspace analysis is always one of the essential tasks in designing any mechanism. However, the unidirectional constraint imposed by cables makes this analysis more challenging for CDRPMs. In the literature, different types of workspace have been introduced based on various definitions for cable-driven parallel manipulators. A number of researchers addressed the set of postures that the end-effector can attain statically while only taking gravity into account [11]. Some researchers addressed a set of postures when cable robots are needed to exert specific wrench to interact with the environment besides maintaining its own static equilibrium. Ebert-Uphoff and Voglewede described this type of workspace as wrench feasible workspace in [12]. Another workspace that has been introduced is called dynamic workspace along with a set of wrenches called pseudo-pyramid. This type of workspace is defined by Barrette and Gosselin as the set of all possible postures of the end-effector of the cable robot with specific acceleration requirement [13].

Finally, one of the most general workspace definitions is referred to the workspace in which any wrench can be generated at the moving platform while cables are in tension. Verhoeven and Hiller termed such workspace as controllable workspace [14]. This kind of workspace depends only on the geometry of the manipulator such as the position of fixed and moving attachment points [15], and therefore, it is important in the conceptual design of such manipulators. Other researchers studied in this type of workspace, including Pham et al. [16] who proposed a "recursive dimension reduction algorithm" to check the closure condition of cable manipulators. Furthermore, McColl and Notash investigated a workspace formulation by using an analytical formulation of the null space of the transposed Jacobian matrix [17]. Null space analysis is the most common method to determine this kind of workspace, which is associated with challenges such as extreme complexity of computations as the degrees of freedom and/or the degrees of redundancy of the robot increase. Also, Gouttefarde and Gosselin addressed the same concept named wrench-closure workspace (WCW), and analytically determined the boundaries of the workspace for planar cable robots [18]. Furthermore, Gouttefarde et al. extended the previous work by investigating the properties of the column of the Jacobian matrix to determine the boundary of constant orientation workspace in [19]. These studies formulated the workspace of manipulators with a complex analytic method. In the literature all the proposed methods including the analytical and numerical methods suffer from a lack of physical interpretation of controllable workspace.

In this paper, the controllable workspace is considered for a general CDRPM with more detailed analysis on the cable manipulator with at least two degrees of redundancy. A set of novel external wrenches called fundamental wrenches is introduced to provide a physical interpretation of this type of workspace. Moreover, an efficient analytic method is developed to determine the controllable workspace boundary of CDRPMs based on fundamental wrenches and rigorous theorems are stated for this method. The proposed method is generally applicable to any cable manipulator with any number of redundant cables as long as its Jacobian matrix is of full rank. Based on this method, a systematic approach is employed to drive

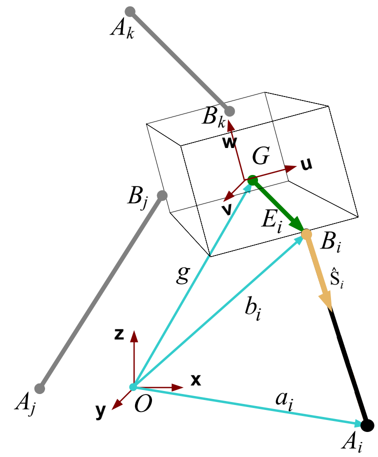


Fig. 1. The general structure of a cable-driven parallel manipulator and its vectors.

the analytic expressions of controllable workspace boundaries. Finally, the proposed method is applied to a spatial manipulator of interest, and conventional numerical methods are used to verify these boundaries.

The remainder of the paper is organized as follows. Definition and characteristics of fundamental wrench is given in Section 2, which is followed by elaboration of an analytic approach to determine the controllable workspace in Section 3. The implementation of the proposed method on the case study is given in Section 4, and Section 5 summarizes the concluding remarks.

2. FUNDAMENTAL WRENCH ANALYSIS

2.1. Background

The general structure of a CDRPM with m cables and n DOF ($m > n$) is shown in Fig. 1. In the figure, $\mathbf{A}_i$ denotes fixed attachment point of the i^{th} cable and $\mathbf{B}_i$ is the point of connection of the i^{th} cable to the end-effector, respectively. Furthermore, $\hat{\mathbf{S}}_i$ is the unit vector connecting point B_i to point A_i, i.e., $\hat{\mathbf{S}}_i = (\vec{a}_i - \vec{b}_i)/||\vec{a}_i - \vec{b}_i||$. The position and the orientation of centre of the moving platform G are denoted by $X = [\mathbf{P}, \Theta] = [x, y, z, \theta_x, \theta_y, \theta_z]^T$, in which, $\mathbf{P}$ and Θ denote the position vector and the orientation of the moving platform with respect to the fixed platform, represented by Euler angles. For the CDRPM, the relationship between the tension force of cables and an external wrench $\mathbf{w}$ applied to centre of the moving platform G is given by [20]:

$$\mathbf{A}\mathbf{f} = \mathbf{w}, \;\; \mathbf{A} = -\mathbf{J}^T \,, \tag{1}$$

where $\mathbf{f} = (f_1, \ldots, f_m)$ is the vector of cable tension force, $\mathbf{w}$ denotes external wrench acting on the moving platform, $\mathbf{A}$ is the structure matrix [21], whose columns are denoted by $\mathbf{A}_i$ and $\mathbf{J}$ is the manipulator Jacobian matrix. In a general 6-DOF CDRPM the wrench vector $\mathbf{A}_i$ is defined as follows [20]:

$$\mathbf{A} = [\mathbf{A}_1, \ldots, \mathbf{A}_m] \;\;,\;\; \mathbf{A}_i = \begin{bmatrix} \hat{\mathbf{S}}_i \\ \mathbf{E}_i \times \hat{\mathbf{S}}_i \end{bmatrix}, \tag{2}$$

in which, as illustrated in Fig. 1, $\mathbf{E}_i$ denotes the position vector of the moving attachment point of the i^{th} cable measured with respect to point G.

To determine the controllable workspace of cable manipulator, Eq. (1) must be solvable for nonnegative cable forces in the presence of any external wrenches at the given configuration. Note that the CDRPM structure matrix **A** is a non-square $n \times m$ matrix. Furthermore, there are many solutions for f to be projected into **w**. For any underdetermined system of equations, the minimal norm solution is defined as follows [22]:

$$\mathbf{f} = \mathbf{A}^{\dagger}\mathbf{w} + \mathbf{N}\mathbf{h}\,, \tag{3}$$

in which, $\mathbf{A}^{\dagger}$ is the pseudo-inverse of structure matrix, **N** denotes a matrix whose columns span the kernel of the matrix **A** and **h** is a vector of arbitrary real numbers.

Definition I: The feasible set is a set of all nonnegative solutions of Eq. (1). It is derived from the general solution Eq. (3) obtained by changing **h** and positive null vector **N**, that satisfy the following condition:

$$\mathbf{F}_{feas} = \{\mathbf{f} \mid \mathbf{f} = \mathbf{A}^{\dagger}\mathbf{w} + \mathbf{N}\mathbf{h}, \mathbf{f} \geq \mathbf{0}\}\,. \tag{4}$$

A set of all manipulator configurations (X) belongs to the controllable workspace, if and only if, the feasible set is nonempty for any wrench exerted on the end-effector of manipulator, that is, a positive null vector of structure matrix exists [14].

2.2. Definition and Characteristics

In this section the concept of fundamental wrench is investigated to give a physical interpretation. Fundamental wrenches are defined as a set of specific external wrenches that depend only on the geometry of the manipulator, shown as follows:

Fundamental Wrench Definition: The fundamental wrench ($\mathbf{w}_f$) is a linear positive combination of r arbitrary column vectors of structure matrix (**A**), in the n DOF cable manipulator with r degree of redundancy and full rank structure matrix. For any arbitrary choice of r components of null vector of the structure matrix, a fundamental wrench is defined as:

$$\mathbf{w}_f = -\mathbf{N}_a\mathbf{A}_a\,, \tag{5}$$

in which $\mathbf{N}_a$ is a vector that contains arbitrary r components of null vector and $\mathbf{A}_a$ denotes a matrix whose columns are r arbitrary columns of structure matrix. By varying the choice of components of the null vector and structure matrix, a set of $(n+r)!/(r!n!)$ fundamental wrenches is obtained.

Note that a set of fundamental wrench exists, if and only if, at least one positive vector of null space of structure matrix exists. This definition is more tractable for manipulator with one degree of redundancy. Fundamental wrench for cable-driven redundant manipulator with one degree of redundancy was introduced previously by the authors in [23] as follows:

$$\mathbf{w}_f = -\mathbf{A}_i \;\;,\;\; i = 1,\ldots,n+1\,. \tag{6}$$

The important property of fundamental wrench is that it is applied on the moving platform, the corresponding cables become ineffective. This claim is rigorously stated in the following theorem.

Theorem 1: If any fundamental wrench of n DOF cable manipulator with r degree of redundancy is applied to the centre of moving platform G with full rank structure matrix (**A**), then feasible nonnegative forces in corresponding cables will be zero at the given configuration.
Proof: According to the fundamental wrench definition, a positive null vector of structure matrix (**A**) exists such that:

$$\mathbf{A}_1 n_1 + \cdots + \mathbf{A}_m n_m = 0 \quad n_i > 0\,. \tag{7}$$

The summation of r arbitrary components of Eq. (7) generates a wrench in the opposite direction of a fundamental wrench. Substitute fundamental wrench to r arbitrary components and move this vector to the right-hand side of Eq. (7) as follows:

$$\mathbf{A}_1 n_1 + \cdots + \mathbf{A}_n n_n = \mathbf{w}_f \,. \tag{8}$$

Because r terms are missing in the left-hand side of Eq. (8), this equation can be interpreted as a case of exerting one of the fundamental wrenches to the end-effector, and no projection of such wrench is seen on the corresponding cables ($\mathbf{A}_{n+1}, \cdots, \mathbf{A}_{n+r}$). This can be physically interpreted by simply removing the r arbitrary cables in such cases or annihilating the r degree of redundancy to perform nonnegative reaction forces in other cables.

Remark 1: According to theorem 1, fundamental wrench is the worst-case wrench that can be generated at the moving platform such that r degrees of redundancy are annihilated at a specific configuration. Therefore, the controllable workspace determined by fundamental wrench is certainly valid for any other typical wrench applied to the moving platform.

The following theorem may be stated based on the definition of fundamental wrench and its physical interpretation for a cable parallel robot to obtain controllable workspace.

Theorem 2: A set of all manipulator configurations (X), belongs to the controllable workspace (CW), if and only if, there exist nonnegative vector forces $\mathbf{f}$ for the cables corresponding to each fundamental wrench applied on the moving platform, such that the force vector $\mathbf{f}$ contains r zero components in the corresponding cables and n positive components in other cables.

$$\forall X \in CW \quad \Longleftrightarrow \quad \mathbf{A}\mathbf{f} = \mathbf{w}_f \;\;,\;\; \mathbf{f} \geq \mathbf{0} \,. \tag{9}$$

Proof: When a typical manipulator configuration X belongs to the controllable workspace, a positive null vector of structure matrix ($\mathbf{A}$) exists [14]. For n DOF cable manipulator with r degree of redundancy, the relation between elements of null vector (n_i) and wrench vectors ($\mathbf{A}_i$) is written as follows:

$$\mathbf{A}_1 n_1 + \cdots + \mathbf{A}_m n_m = \mathbf{0} \;\;,\;\; n_i > 0 \;\;,\;\; m = n + r \,. \tag{10}$$

Considering the fundamental wrench definition, move r elements from the left-hand side to the right-hand side and write the resulting equation as follows:

$$\mathbf{A}_1 n_1 + \cdots + \mathbf{A}_n n_n = \mathbf{w}_f \;\;,\;\; \mathbf{w}_f = -\mathbf{A}_{n+1} n_{n+1} - \cdots - \mathbf{A}_m n_m \,. \tag{11}$$

According to Eq. (11), a nonnegative solution $[n_1, \cdots, n_n, 0, \cdots, 0]^T$ with r zero elements exists in the corresponding cables for fundamental wrench. This completes the proof of necessary condition.

To prove sufficient condition, assume that there exists nonnegative solution $\mathbf{f}$ with r zero components, such that:

$$\mathbf{A}\mathbf{f} = -\mathbf{A}_{n+1} n_{n+1} - \cdots - \mathbf{A}_{n+r} n_{n+r}, \;\; \mathbf{f} = [f_1, \ldots, f_n, 0, \ldots, 0], \;\; f_i > 0 \,. \tag{12}$$

It is sufficient to show that the positive combination of wrench vectors is obtained by moving the corresponding right-hand side terms to the left-hand side as follows:

$$\mathbf{A}_1 f_1 + \cdots + \mathbf{A}_n f_n + \mathbf{A}_{n+1} n_{n+1} + \cdots + \mathbf{A}_{n+r} n_{n+r} = 0 \,. \tag{13}$$

The tension force vector ($[f_1, \ldots, f_n, n_{n+1}, \ldots, n_{n+r}]^T$) is a positive vector belonging to the null space of structure matrix ($\mathbf{A}$). Hence, a typical manipulator configuration X belongs to the controllable workspace. According to the theorem 2, the controllable workspace can be determined only by investigation of the existence of solution to $\mathbf{A}\mathbf{f} = \mathbf{w}_f$ for each fundamental wrench.

3. CONTROLLABLE WORKSPACE ANALYSIS

One of the distinctive advantages of introducing fundamental wrench is its physical interpretation that relates it to zero tension force for each cable, and therefore, this investigation is reduced to looking for existence of such wrenches in the workspace. An analytical method is described in the following sections to search for controllable workspace boundaries based on linear algebra. Let us first introduce the analytical method to obtain controllable workspace for cable manipulators with one degree of redundancy. Then, this method is extended for cable manipulators with more than one degree of redundancy.

3.1. n DOF with one degree of redundancy

By considering theorem 2, the controllable workspace of manipulator with one degree of redundancy may be obtained by nonnegative solution to:

$$\mathbf{A}_{n\times(n+1)}\mathbf{f}_{n+1} = \mathbf{w}_f, \;\; \mathbf{w}_f = -\mathbf{A}_i\,. \tag{14}$$

Considering the fundamental wrench definition, move r elements from the left-hand side to the right-hand side and write the resulting equation as follows:

$$[\mathbf{A}_1 \;\; \ldots \;\; \mathbf{A}_{n+1}]\,[f_1 \;\; \ldots \;\; f_{i-1} \;\; 0 \;\; f_{i+1} \;\; \ldots \;\; f_{n+1}]^T = \mathbf{w}_f \;\; , \;\; \mathbf{w}_f = -\mathbf{A}_i\,. \tag{15}$$

It is easy to show that $\mathbf{A}_i$ can be removed from the left-hand side of Eq. (15) and the remaining equation may be represented as a function of unknowns:

$$[\mathbf{A}_1 \;\; \ldots \;\; \mathbf{A}_{i-1} \;\; \mathbf{A}_{i+1} \;\; \ldots \; \mathbf{A}_{n+1}]\,[f_1 \;\; \ldots \;\; f_{i-1} \;\; f_{i+1} \;\; \ldots \;\; f_{n+1}]^T = -\mathbf{A}_i\,. \tag{16}$$

From concepts of linear algebra the analytic solution of Eq. (16) can be found by:

$$f_{ji} = \frac{\Delta_{ji}}{\Delta_i} = \frac{det([\mathbf{A}_1 \;\; \ldots \;\; \mathbf{A}_{j-1} \;\; -\mathbf{A}_i \;\; \mathbf{A}_{j+1} \;\; \ldots \;\; \mathbf{A}_{i-1} \;\; \mathbf{A}_{i+1} \;\; \ldots \;\; \mathbf{A}_{n+1}])}{det([\mathbf{A}_1 \;\; \ldots \;\; \mathbf{A}_{i-1} \;\; \mathbf{A}_{i+1} \;\; \ldots \;\; \mathbf{A}_{n+1}])}\,, \tag{17}$$

in which, f_{ji} is the j^{th} component of $[f_1 \;\; \ldots \;\; f_{i-1} \;\; f_{i+1} \;\; \ldots \;\; f_{n+1}]^T$ when the i^{th} fundamental wrench is exerted on the moving platform, Δ_i denotes the determinant of structure matrix without i^{th} column and Δ_{ji} is the determinant of structure matrix without i^{th} column, while its j^{th} column is $\mathbf{A}_i$.

Based on theorem 2, the manipulator configuration (X) belongs to the controllable workspace when all f_{ji} become positive for each fundamental wrench. Therefore, Δ_{ji} and Δ_i must be the same sign for each f_{ji}. Hence, $(n+1)$ equations of Δ_{ji} and Δ_i exist by which they have a same sign. Thus, $n\times(n+1)$ relations exist for all fundamental wrenches. These relations possess an interesting property that $(n+1)$ relations obtained from each fundamental wrench are equivalent to $(n+1)$ equations obtained from other fundamental wrenches by displacement columns as follows:

$$\begin{aligned}
\Delta_{ji} &= \det[\mathbf{A}_1 \;\cdots\; \mathbf{A}_{j-1} \;\; -\mathbf{A}_i \;\; \mathbf{A}_{j+1} \;\cdots\; \mathbf{A}_{i-1} \;\; \mathbf{A}_{i+1} \;\cdots\; \mathbf{A}_n] \\
&= -\det[\mathbf{A}_1 \;\cdots\; \mathbf{A}_{j-1} \;\; \mathbf{A}_i \;\; \mathbf{A}_{j+1} \;\cdots\; \mathbf{A}_{i-1} \;\; \mathbf{A}_{i+1} \;\cdots\; \mathbf{A}_n] \\
&= \pm\det[\mathbf{A}_1 \;\cdots\; \mathbf{A}_{j-1} \;\; \mathbf{A}_{j+1} \;\cdots\; \mathbf{A}_{i-1} \;\; \mathbf{A}_i \;\; \mathbf{A}_{i+1} \;\cdots\; \mathbf{A}_n] \\
&= \pm\det[\mathbf{A}_1 \;\cdots\; \mathbf{A}_{j-1} \;\; \mathbf{A}_{j+1} \;\cdots\; \mathbf{A}_n] \\
&= \pm\Delta_j\,,
\end{aligned} \tag{18}$$

where, the sign $\pm$ depends on the sign of the permutation accomplished to place $\mathbf{A}_i$ between $\mathbf{A}_{i-1}$ and $\mathbf{A}_{i+1}$. Accordingly, the enclosed region of controllable workspace of one degree of redundancy is obtained by applying an arbitrary fundamental wrench to the end-effector. Moreover, the closed form expression of boundaries of this region may be obtained in which each $(n+1)$ equation become zero. In this way, McColl and Notash [17] presented a similar workspace formulation, but their method does not cover all enclosed regions of whole controllable workspace with more than one degree of redundancy. Furthermore, their method is not accompanied by any mathematical proof.

3.2. n DOF with more than one degree of redundancy

For workspace analysis in this general case, the manipulator is divided into two categories based on structure matrix. One of them is a sub-robot, introduced in [23], which is part of the robot with only one degree of redundancy. Therefore, the structure matrix of sub-robot ($\mathbf{A}_{sub}$) is obtained by arbitrarily choosing $n+1$ columns of the structure matrix of the original manipulator. The part of controllable workspace of the original manipulator is obtained by the union of controllable workspaces of all sub robots [23], but all regions of the controllable workspace of the main manipulator are not covered by the union of controllable workspace of all sub-robots. In this case, the effect of the remaining $(r-1)$ redundant actuators is not considered. This concept of the enclosed region of controllable workspace of CDRPMs with more than one degree of redundancy is an important issue that is considered in this paper while other researchers such as Gouttefarde et al. [19] have not discussed this in their studies. For illustration, consider the case where the fundamental wrench of one sub-robot is generated by h cables while $h < n$. In this case, more than one cable is slack. Furthermore, other sub-robots exist such that their fundamental wrenches are produced by k cables with $k \leq n-h$. This concept represents an important characteristic of controllable workspace of cable CDRPMs with more than one degree of redundancy. Thus, the combination of fundamental wrenches of these sub-robots generates a fundamental wrench by n cables in the fully constrained configuration. Therefore, analysis of the combination of sub-robot is necessary.

Combined Sub-robot Definition: Combined sub-robot represents a manipulator with a combined structure matrix as follows:

$$\mathbf{A}_{Csub} = [\mathbf{A}_s \mid \mathbf{A}_c]_{n\times(n+1)}\,, \tag{19}$$

in which, $\mathbf{A}_s$ is an arbitrary n column of the main manipulator structure matrix and $\mathbf{A}_c$ is positive linear combination of other k columns, where $2 \leq k \leq r$. The combined column vector $\mathbf{A}_c$ is obtained from the following equation:

$$\mathbf{A}_c = \sum_k \mathbf{A}_k,\ \ 2 \leq k \leq r\,. \tag{20}$$

For each sub-robot (or combined sub-robot), Eq. (21) (or Eq. 22) results in $n+1$ vector relationships (see Eq. (14)).

$$\sum_{i=1, i\neq j}^{n+1} \mathbf{A}_{sub_i} f_{i_j} = -\mathbf{A}_{sub_j},\ \ j = 1,\ldots,n+1\,, \tag{21}$$

$$\sum_{i=1, i\neq j}^{n+1} \mathbf{A}_{Csub_i} f_{i_j} = -\mathbf{A}_{Csub_j},\ \ j = 1,\ldots,n+1\,. \tag{22}$$

Controllable workspace of the main manipulator is obtained from the union of controllable workspace of sub-robots and combined sub-robots. Furthermore, the following efficient theorem may be stated based on fundamental wrench definition and sub-robot and combined sub-robot to obtain controllable workspace.

Theorem 3: A set of all manipulator configurations X belongs to the controllable workspace with full rank structure matrix, if and only if, it belongs to the union of controllable workspaces of all possible sub-robots and combined sub-robots.

Proof: Assume that a typical manipulator configuration X belongs to the union of controllable workspaces of sub-robots and combined sub-robots. Thus, there exists at least one positive solution of Eq. (21) from sub-robots or Eq. (21) from combined sub-robots. First, assume that there exists a positive solution for at least one sub-robot. Without loss of generality, assume that $\mathbf{A}_1, \cdots, \mathbf{A}_n$ is linear independent. The structure matrix of each sub-robot is defined as follows:

$$\mathbf{A}_{sub} = [\mathbf{A}_1 \ \ \cdots \ \ \mathbf{A}_n \ \ \mathbf{A}_{n+j}], \ \ j = 1, \ldots, r. \tag{23}$$

Also, assume that there exists a positive solution for a sub-robot with $j = 1$. By this assumption, Eq. (21) may be represented by the following equation:

$$\sum_{i=1}^{n} \mathbf{A}_i f_{i_{n+1}} = -\mathbf{A}_{n+1}, \ \ f_{i_{n+1}} > 0. \tag{24}$$

Furthermore, $(r-1)$ equations exist for $j = 2, \cdots, r$ so that they may be written as follows:

$$\sum_{i=1}^{n} \mathbf{A}_i f_{i_{n+j}} = -\mathbf{A}_{n+j}, \ \ 2 \le j \le r. \tag{25}$$

In this case, multiply a positive coefficient (c) to Eq. (24) and sum the result to Eq. (25) for all possible j and write the result as:

$$c\sum_{i=1}^{n} \mathbf{A}_i f_{i_{n+1}} + \sum_{i=1}^{n} \mathbf{A}_i f_{i_{n+2}} + \cdots + \sum_{i=1}^{n} \mathbf{A}_i f_{i_{n+r}} = -c\mathbf{A}_{n+1} - \mathbf{A}_{n+2} - \cdots - \mathbf{A}_{n+r}. \tag{26}$$

By moving the right-hand side terms to the left, Eq. (26) can be represented as follows:

$$\mathbf{A}_1 f_1 + \cdots + \mathbf{A}_n f_n + c\mathbf{A}_{n+1} + \mathbf{A}_{n+2} + \cdots + \mathbf{A}_{n+r} = 0, \tag{27}$$

where,

$$f_i = (c f_{i_{n+1}} + f_{i_{n+2}} + \cdots + f_{i_{n+r}}), \ \ i = 1, \ldots, n. \tag{28}$$

In Eq. (27), a positive coefficient (c) can be found that $f_1, \cdots, f_n$ become positive. Therefore, the resulting vector $[f_1, \cdots, f_n, c, 1, \cdots, 1]$ is a positive vector belonging to the null space of structure matrix ($\mathbf{A}$). Therefore, the manipulator configuration X belongs to the controllable workspace when it belongs to the controllable workspace of each sub-robot.

Now, assume that a typical manipulator configuration X belongs to the controllable workspace of combined sub-robot. The combined wrench $\mathbf{A}_{Csub}$ is divided into two parts. For the first part, assume that a positive solution exists for Eq. (22) where $\mathbf{A}_c$ is the positive linear combination of r wrenches for the combined structure matrix as follows:

$$\mathbf{A}_{Csub} = [\mathbf{A}_1 \ \ \cdots \ \ \mathbf{A}_n \ \ \sum_{j=1}^{r} \mathbf{A}_{n+j}]. \tag{29}$$

By this assumption, write Eq. (22) and move the right-hand side terms to the left. The resulting equation is obtained by:

$$\mathbf{A}_1 f_1 + \cdots + \mathbf{A}_n f_n + \mathbf{A}_{n+1} + \cdots + \mathbf{A}_{n+r} = 0, \ \ f_i > 0. \tag{30}$$

The vector $[f_1, \cdots, f_n, 1, \cdots, 1]$ is the positive null vector of structure matrix ($\mathbf{A}$). Therefore, the manipulator configuration X belongs to the controllable workspace of the major manipulator. For the other part, assume a positive solution exists for Eq. (22), where $\mathbf{A}_c$ is the positive linear combination of k wrenches ($k < r$), for combined structure matrix as follows:

$$\mathbf{A}_{Csub} = [\mathbf{A}_1 \;\; \cdots \;\; \mathbf{A}_n \;\; \sum_{j=1}^{k<r} \mathbf{A}_{n+j}] \,. \tag{31}$$

By this assumption, Eq. (22) is given as:

$$\mathbf{A}_1 f_1 + \cdots + \mathbf{A}_n f_n = -\mathbf{A}_{n+1} - \cdots - \mathbf{A}_{n+r}, \;\; f_i > 0 \,. \tag{32}$$

Also, there exist $(r-k)$ sub-robots of Eq. (21) with $\mathbf{A}_{sub_j} = \mathbf{A}_{n+k+1}, \ldots, \mathbf{A}_{n+r}$ that may be written as follows:

$$\sum_{i}^{n} \mathbf{A}_i f_{i_j} = -\mathbf{A}_j, \;\; j = n+k+1, \ldots, n+r \,. \tag{33}$$

It can be shown that at least one positive combination of structure matrix wrenches exists when we multiply a positive coefficient (c) to Eq. (32) and add to Eq. (33) for all possible $\mathbf{A}_j$.

$$c(\mathbf{A}_1 f_1 + \cdots + \mathbf{A}_n f_n) + \sum_{i}^{n} \mathbf{A}_i f_{i_{n+k+1}} + \cdots + \sum_{i}^{n} \mathbf{A}_i f_{i_{n+r}} = -c(\mathbf{A}_{n+1} + \cdots + \mathbf{A}_{n+k}) - \mathbf{A}_{n+k+1} - \cdots - \mathbf{A}_{n+r} \,. \tag{34}$$

Similar to the result of Eqs. (26,27), this combination may be written as follows:

$$\mathbf{A}_1 f_{c_1} + \cdots + \mathbf{A}_n f_{c_n} + c\mathbf{A}_{n+1} + \cdots + c\mathbf{A}_{n+k} + \mathbf{A}_{n+k+1} + \cdots + \mathbf{A}_{n+r} = 0 \,, \tag{35}$$

where,

$$f_{c_i} = c f_i + f_{i_{n+k+1}} + \cdots + f_{i_{n+r}} \,. \tag{36}$$

The positive vector $[f_{c1} \;\; \cdots \;\; f_{cn} \;\; c \;\; \cdots \;\; c \;\; 1 \;\; \cdots \;\; 1]$ belongs to the null space of structure matrix ($\mathbf{A}$). Therefore, the manipulator configuration (X) belongs to the controllable workspace when it belongs to the controllable workspace of each combined sub-robot.

To prove the necessary condition, assume that a typical manipulator configuration (X) belongs to the controllable workspace. In such cases a positive vector ($[n_1 \;\; \cdots \;\; n_{n+r}]$) belonging to the null space of ($\mathbf{A}$) exists such that:

$$\sum_{1}^{n+r} \mathbf{A}_i n_i = 0, \;\; n_i > 0 \,. \tag{37}$$

By dividing the equation to n_j and moving $\mathbf{A}_j$ to the right side, the resulting equation can be written as follows:

$$\mathbf{A}_1 m_{j1} + \cdots + \mathbf{A}_{j-1} d_{j(j-1)} + \mathbf{A}_{j+1} d_{j(j+1)} + \cdots + \mathbf{A}_{n+r} d_{j(n+r)} = -\mathbf{A}_j, \;\; d_{ji} = \frac{n_i}{n_j} \,, \tag{38}$$

in which, $[d_{j1}, \cdots, d_{j(j-1)}, d_{j(j+1)}, \cdots, d_{j(n+r)}]$ is a positive vector, and therefore, it is a feasible solution for Eq. (38).

According to the fundamental theorem of linear programming, if a feasible solution exists, then a basic feasible solution can be found [24]. A basic solution can be obtained by setting n-m variables to zero and solving the following constraint equation simultaneously [25].

$$\mathbf{A}_{n\times m} \mathbf{x}_m = \mathbf{B}_n, \;\; \mathbf{x} \geq \mathbf{0}, \; m > n \,. \tag{39}$$

Therefore, the basic feasible solutions may be found for Eq. (38). Without loss of generality, assume that $\mathbf{A}_1 \ldots \mathbf{A}_n$ are linearly independent vectors and $[b_{j1} \ldots b_{jn}]^T$ are a basic feasible solution so that:

$$\sum_{i=1}^{n} \mathbf{A}_i b_{ji} = -\mathbf{A}_j, b_{ji} \geq 0 . \tag{40}$$

When at least one of the basic feasible solutions $[b_{j1} \ldots b_{jn}]^T$ is strictly positive, the manipulator configuration X belongs to the controllable workspace of sub-robots based on theorem 2. Assume that the basic feasible solutions are nonnegative, that is, there exist zero elements in basic solution. In the worst possible case, one element is strictly positive and the other elements are zero. Without loss of generality, assume that $b_{j1} > 0$ and $b_{j2} = \cdots = b_{jn} = 0 \;\; j = n+1, \cdots, n+r-1$ and simplify Eq. (40) as follows:

$$\mathbf{A}_1 b_{j1} = -\mathbf{A}_j, \;\; j = n+1, \ldots, n+r-1 . \tag{41}$$

Similar to Eq. (38), divide Eq. (37) to the component n_{n+r} and move $\mathbf{A}_{n+r}$ to the right-hand side, the resulting equation can be written as follows:

$$\mathbf{A}_1 d_{(n+r)1} + \cdots + \mathbf{A}_{n+r-1} d_{(n+r)(n+r-1)} = -\mathbf{A}_{n+r} , \tag{42}$$

in which, $[d_{(n+r)1}, \cdots, d_{(n+r)(n+r-1)}]$ is a feasible solution. Therefore, a basic feasible solution exists. For all $\mathbf{A}_j (j = n+1, \cdots, n+r-1)$, substitute Eq. (41) in Eq. (42) and write the result as:

$$\mathbf{A}_1 (d_{(n+r)1} - \sum_{j=n+1}^{n+r} d_{(n+r)j} b_{j1}) + \mathbf{A}_2 d_{(n+r)2} + \cdots + \mathbf{A}_n d_{(n+r)n} = -\mathbf{A}_{n+r} . \tag{43}$$

The vector $[(d_{(n+r)1} - \sum_{j=n+1}^{n+r} d_{(n+r)j} b_{j1}), m_{(n+r)2}, \ldots, m_{(n+r)n}]^T$ is nonnegative and forms the basic feasible solution with $n-1$ strictly positive elements $d_{(n+r)2}, \ldots, d_{(n+r)n}$. If the remaining elements $(d_{(n+r)1} - \sum_{j=n+1}^{n+r} d_{(n+r)j} b_{j1})$ are strictly positive, the basic feasible solution is strictly positive, therefore, manipulator configuration (X) belongs to the controllable workspace of sub-robot. Assume that $(d_{(n+r)1} - \sum_{j=n+1}^{n+r} d_{(n+r)j}$ $b_{j1}) = 0$ and write the resulting equation as:

$$\mathbf{A}_2 d_{(n+r)2} + \cdots + \mathbf{A}_n d_{(n+r)n} = -\mathbf{A}_{n+r} . \tag{44}$$

In Eq. (44), the wrench vector $\mathbf{A}_{n+r}$ depends on wrench vectors $\mathbf{A}_2, \cdots, \mathbf{A}_n$. by adding Eq. (41) for arbitrary $\mathbf{A}_j (j = n+1, \cdots, n+r-1)$ to Eq. (44), hence the resulting equation can be obtained from the following equation:

$$\mathbf{A}_1 b_{j1} + \mathbf{A}_2 d_{(n+r)2} + \cdots + \mathbf{A}_n d_{(n+r)n} = -\mathbf{A}_j - \mathbf{A}_{n+r} . \tag{45}$$

Note that the vector $[b_{j1}, d_{(n+r)2}, \cdots, d_{(n+r)n}]^T$ is strictly positive. Therefore, the manipulator configuration X belongs to the controllable workspace of the combined sub-robot. This completes the proof of the necessary part of the theorem.

4. CASE STUDY

In this section the proposed method is used to determine the controllable workspace of a spatial CDRPM. KNTU CDRPM is six degrees of freedom fully constrained cable manipulator actuated by eight cables. This manipulator is under investigation for possible high speed and wide workspace applications at K. N. Toosi

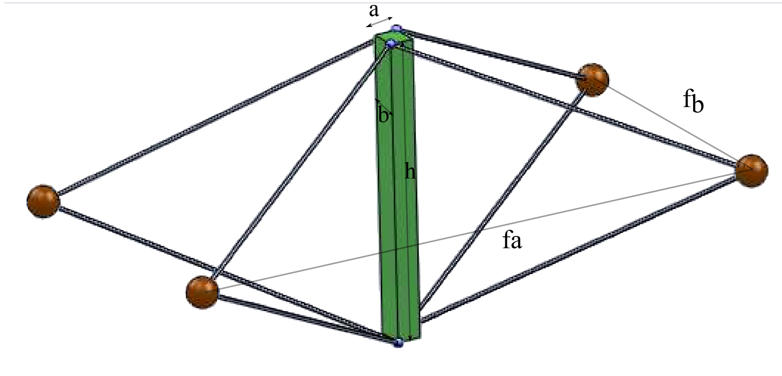


Fig. 2. The KNTU CDRPM Galaxy design.

Description	Value
fa: fixed Base's length	4 m
fb: fixed Base's width	2 m
a: moving platform's length	0.28 m
b: moving platform's width	0.14 m
h: moving platform's height	1 m

Table 1. Geometric parameters of Galaxy design.

University (KNTU). There exist different designs for KNTU CDRPM based on different approach such as collision avoidance scheme, force feasibility, and dexterity. A special design of KNTU CDRPM is shown in Fig. 2, which is called Galaxy.

In this design the fixed and moving attachment points are carefully located at suitable locations to increase the rotation workspace of the robot. As shown in Fig.2, the fixed and moving attachment points are coincident at one point in pairs, whose geometric parameters are given in Table 1 for simulations.

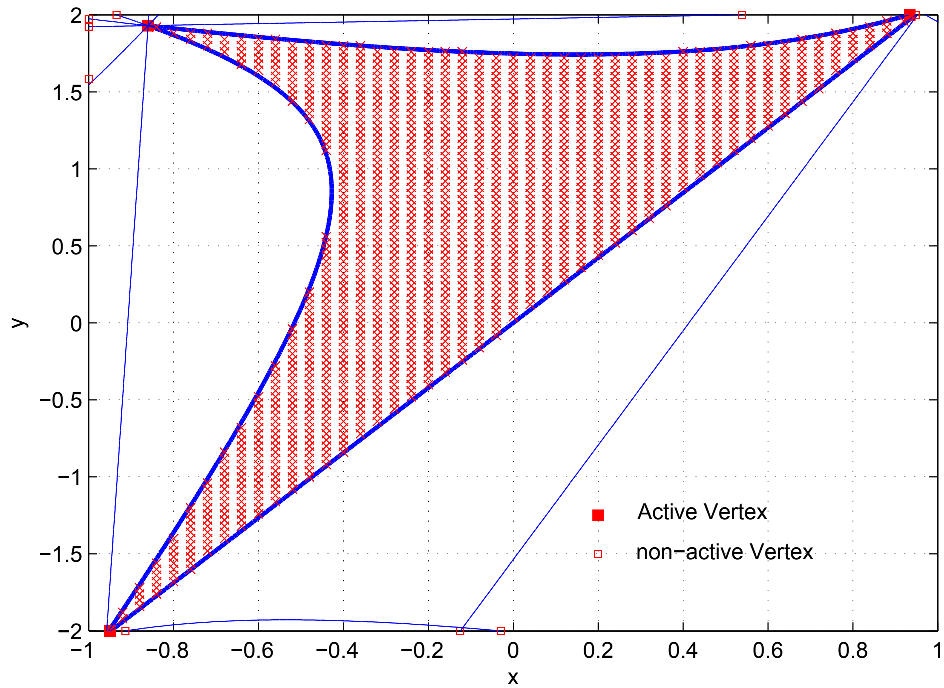
To determine the controllable workspace of cable redundant manipulator, controllable workspace of each sub-robot and combined sub-robot are obtained. According to the proposed method, the union of these workspaces forms the overall workspace. Moreover, each of them indicates the effect of the corresponding annihilated cables in the controllable workspace of the main manipulator. This physical interpretation may be applicable for optimal fault tolerant design of such manipulators. As an example consider the boundaries illustrated in Fig. 3 . In this figure the controllable workspace of first sub-robot by removing the first cable is significantly greater than that of the fifth sub-robot.

To obtain the boundary of controllable workspace that each sub-robot obtains, the structure matrix of the sub-robot is determined by using the proposed method. Without loss of generality, the structure matrix of the first sub-robot is obtained from:

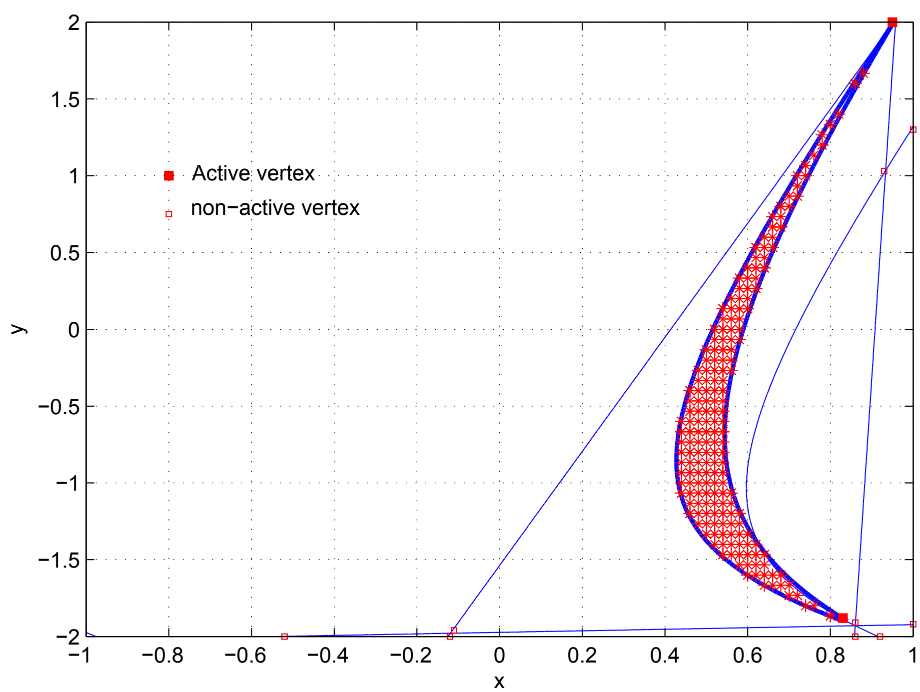
$$\mathbf{A}_{sub} = [\mathbf{A}_2 \ \cdots \ \mathbf{A}_8]_{6\times 7} \, . \tag{46}$$

Other structure matrix of the sub-robot is similar to Eq. (46) with different wrench vector in the corresponding cable. After finding the structure matrix of the sub-robot, the controllable workspace is obtained based on the proposed method. The related controllable workspace equation for 1^{th} sub-robot and its Δ_j can be written as follows:

$$\sum_{i=3}^{8} \mathbf{A}_i f_{s_i} = -\mathbf{A}_2 \ \ , \ \ \Delta_j = |\mathbf{A}_3 \cdots \mathbf{A}_{j-1} \ \ -\mathbf{A}_2 \ \ \mathbf{A}_{j+1} \cdots \mathbf{A}_8| \, . \tag{47}$$



(a)



(b)

Fig. 3. (a) The curves plotted and controllable workspace for 1^{th} sub-robot (b) for 5^{th} sub-robot, $([\alpha, \beta, \gamma] = [0, 0, 0])$ and $(z = 0.15)$.

An important property of determinants is that they depend linearly on any of their columns. This property can be used to expand the determinant and simplify its expression [26]. This analytical expression for Δ_j is obtained from the following equation:

$$\Delta_j = \phi_1 x^3 + \phi_2 x^2 y + \phi_3 x^2 z + \phi_4 xyz + \cdots + \phi_{19} z + \phi_{20}\,, \tag{48}$$

where ϕ_i's are defined in terms of determinants as a function of orientation coordinates and architecture parameters [26]. As described in the previous section, the boundaries of workspace include some of the curves (Δ_j). Although these curves are obtained in an implicit form, the intersection points between the curves are obtained using a numerical routine in MATLAB. An active intersection point is the one which lies on the vertex of the boundaries of the workspace. Hence, all intersection points are checked to extract the active ones. For active intersection points, the Δ_is have the same sign or are equal to zero. Therefore, the curves that lie between two active intersection points form the boundary of workspace. By this means and unlike other methods such as [19], all variations of workspaces introduced in [27] such as constant orientation or constant position workspace can be suitably determined by the proposed method.

To determine the boundary of combined sub-robot, a similar approach is used. For this case, the equations for the combined sub-robot by removing the first and second cables are given by:

$$\sum_{i=3}^{8} \mathbf{A}_i f_{cs_i} = -\mathbf{A}_1 - \mathbf{A}_2 \;\;,\;\; \Delta_i = |\mathbf{A}_3 \cdots \mathbf{A}_{i-1} \;\; (-\mathbf{A}_1 - \mathbf{A}_2) \;\; \mathbf{A}_{i+1} \cdots \mathbf{A}_8|\,. \tag{49}$$

According to the linear decomposition of determinant [26], equation of Δ_i can be expanded to the following equation:

$$\Delta_i = |\mathbf{A}_3 \cdots \mathbf{A}_{i-1} \;\; -\mathbf{A}_1 \;\; \mathbf{A}_{i+1} \cdots \mathbf{A}_8| + |\mathbf{A}_3 \cdots \mathbf{A}_{i-1} \;\; -\mathbf{A}_2 \;\; \mathbf{A}_{i+1} \cdots \mathbf{A}_8|\,. \tag{50}$$

Similar to sub-robot, the equation expressions of combined sub-robot have a maximum degree of three, while the total degree of the terms in the polynomial expression is also three. The controllable workspace for the combined sub-robot by removing the first and second cables are shown in Fig. 4, for the constant orientation $[\alpha, \beta, \gamma] = [0, 0, 0]$) and constant position along the z axis $(z = 0.15)$. In these figures, the active intersection points and boundary curves are displayed.

Figure 5 shows the controllable workspace of KNTU CDPM due to the union of controllable workspace of its sub-robots and combined sub-robots, for constant orientation. In this figure, all workspaces of sub-robots and combined sub-robots are shown. Moreover, a 2D cross-section of constant orientation controllable workspace of this robot is shown in Fig. 6 for zero orientation in all three Euler angles. It is observed that the controllable workspace of KNTU CDRPM is more than 40% of the whole workspace. In this analysis, we also considered the cables collision and exclude them from constant orientation controllable workspace. The cable collision is determined according to the algorithm given in [28]. Table 2 summarizes the comparison results between various constant orientations of two different types of KNTU CDRPM designs, namely Galaxy and Neuron [28], in which the Neuron design is the first conceptual design given for this structure, while Galaxy design is developed after careful examination of this manipulator to increase the orientation workspace. In the first column of Table 2, several fixed orientations are considered, while in the second and third columns, the percentage of constant orientation controllable workspace is determined for Neuron, and Galaxy design. As observed from the results given in this table, the controllable workspace of Neuron design is limited to only less than 7% of the whole workspace. Also, the controllable workspace is significantly increased to more than 40% in Galaxy design for commonly used orientation. However, this percentage for Galaxy design is greater than Neuron design even in specific higher orientation requests. The Galaxy design may be further optimized by optimization schemes to increase the controllable workspace in more orientation. In these optimization schemes, the analytical solution of the controllable workspace presented in this paper can be extensively used to find optimal configurations.

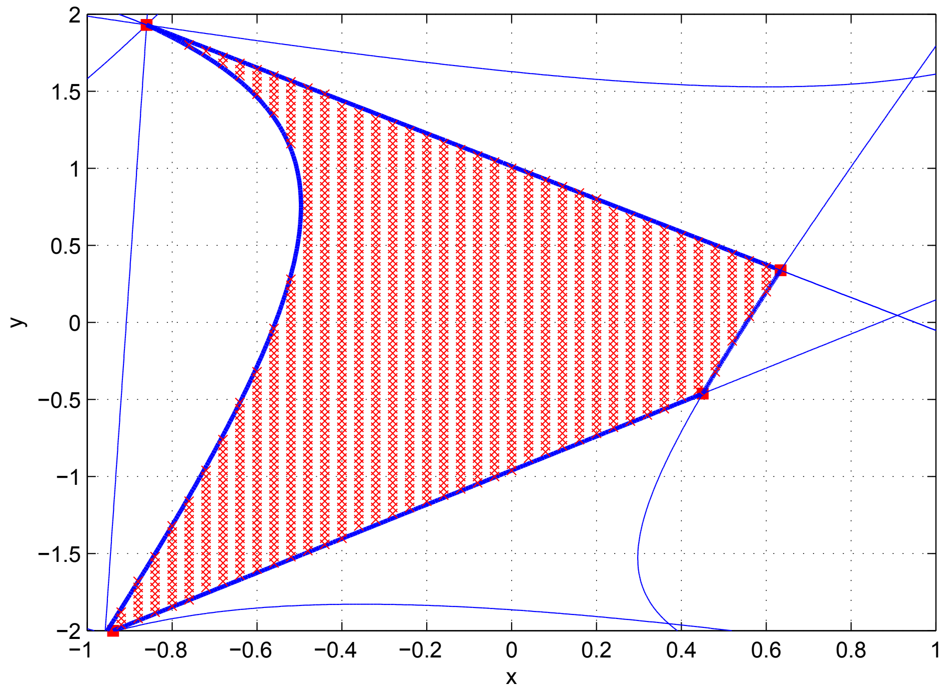


Fig. 4. The curves plotted for $1, 2^{nd}$ combined sub-robot, ($[\alpha, \beta, \gamma] = [0, 0, 0]$) and ($z = 0.15$).

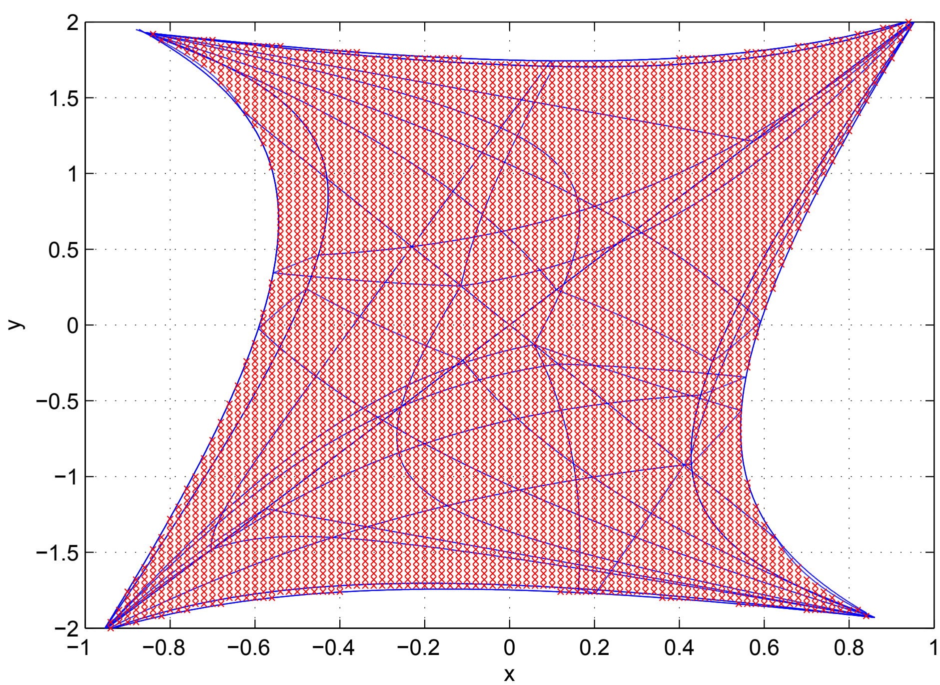
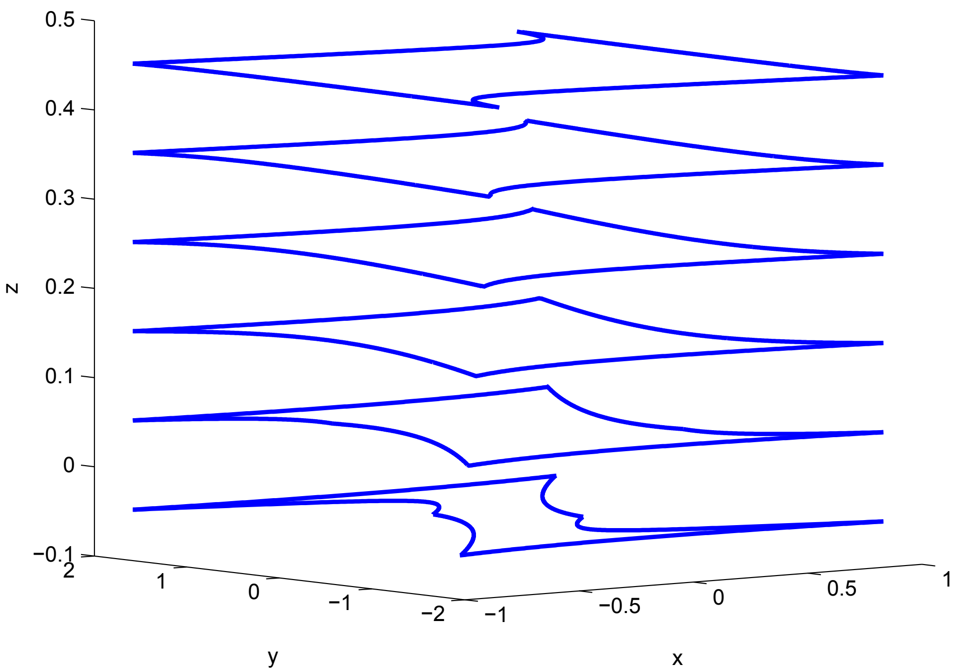
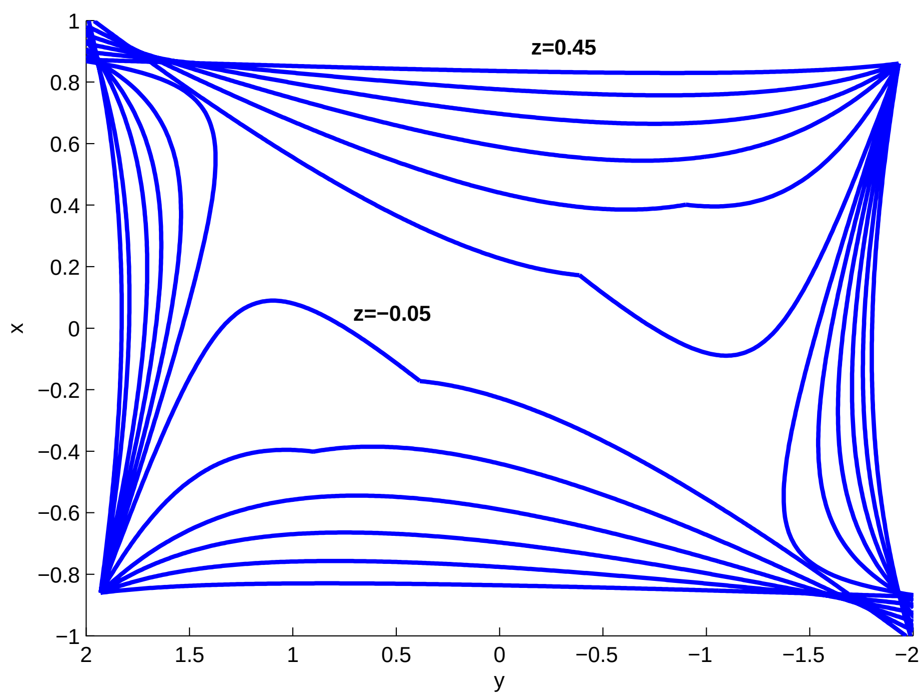


Fig. 5. Union of controllable workspace of sub-robots and combined sub-robots, ($[\alpha, \beta, \gamma] = [0, 0, 0]$) and ($z = 0.15$).



(a)



(b)

Fig. 6. (a) Constant orientation controllable workspace of KNTU CDRPM (b) Contour of constant orientation controllable workspace, ($[\alpha, \beta, \gamma] = [0, 0, 0]$).

Orientation (degrees)	Neuron	Galaxy
(0,0,0)	5.8%	45.4%
(10,0,0)	6.1%	43.5%
(0,10,0)	5.9%	42.9%
(0,0,10)	5.9%	41.3%
(30,30,30)	4.3%	12.6%
(45,45,45)	4.3%	5.2%

Table 2. Percentage of constant orientation controllable workspace.

5. CONCLUSIONS

This paper introduces a set of newly defined fundamental wrenches for the analysis of controllable workspace of cable-driven parallel manipulators. Using this definition a physical interpretation of controllable workspace may be given and the complexity of controllable workspace analysis may be significantly reduced. At a given configuration, fundamental wrench is defined as the worst possible exerted wrench on the moving platform, through which r degree of redundancy is annihilated in the cable manipulator with r degree of redundancy. Through the given theorems developed in this paper, careful examination of the effect of exerting such wrenches on the moving platform, and their physical interpretation is elaborated. Moreover, a systematic method is developed to determine the controllable workspace of redundant cable-driven parallel manipulators based on fundamental wrench. The proposed method is generally applicable to any cable manipulator with any redundant actuation as long as its Jacobian matrix is of full rank. Linear algebra is employed to determine the boundary of controllable workspace for such manipulators in an implicit manner. This method is applied to a spatial manipulator as a case study, and the boundary of controllable workspace is determined for that manipulator. Due to the physical interpretation that this approach adds to the analysis of controllable workspace, it is believed that this representation can be further used as the basis of multi-objective optimization routines including increasing of controllable workspace of cable-driven manipulators.

ACKNOWLEDGEMENTS

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