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# Closed-Form Dynamic Formulation of Spherical Parallel Manipulators by Gibbs-Appell Method

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***Abstract*—Spherical Parallel Robot (SPR) is a complex but widely used type of manipulators that performs only rotational motion. Dynamic analysis of SPR has a vital role in mechanical design, model-based controller, identification and fault detection of such robots. Complexity of SPR kinematic structure makes traditional dynamic modeling methods such as Newton-Euler, virtual work and Lagrange formulations a prohibitive task. In this paper a new procedure for deriving closed form dynamics of general SPR using Gibbs-Appell method is presented. The proposed method does not require any recursive computation or symbolic manipulation and dynamic matrices of the robot is directly derived in an explicit form. By using the proposed method, closed form dynamic formulation of a general 3DOF SPR, known as agile wrist, is obtained and it is verified for an arbitrary trajectory. The unique feature of the method presented in this paper, makes it promising to be used for other parallel manipulators.**

***Keywords*— *spherical parallel robots, dynamic formulation, Gibbs-Appell, closed form formulation, Agile-Wrist***

## I. Introduction

Parallel robot is a kind of mechanism which consists of closed kinematic chains, that connects the moving platform to the robot base. There are various designs for parallel robots having different degrees of freedom which are combination of rotational and translational displacement. The most well known parallel robot is Stewart–Gough platform that has 6 DOFs [1]. For some applications it is unnecessary to have all possible DOFs, thus having a mechanism with fewer DOFs that are optimized for a specific task is of practical interest.

Spherical parallel robot (SPR) is a kind of mechanism in which all the links have pure rotational motion. Several SPRs with different architectures have been reported in the literature; and as a representative one may look into [2], [3]. Some applications of SPR are in minimally invasive surgery [4], trust control [5], and rehabilitation. A typical example of SPR, named, Agile-Wrist is shown in Fig. 1, in which the moving platform of the robot has three pure rotational degrees of freedom. In [6] kinematic, workspace and singularity analyses of the robot are investigated . Furthermore, an structure optimization is performed in the design of such robot in [7].

Closed kinematic chains and variety of passive joints makes dynamic modeling of parallel robot difficult. In the last two decades more attention have been given on kinematic modeling, while work on general formulation for dynamic problem have been less reported in research field [8]. Whereas, having a closed form dynamic model for robot in the form of:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Q \tag{1}$$

is very applicable in mechanism design and its optimization, controller design as well as parameters identification and fault detection. [9] proves that there is a closed form solution for any natural mechanism. Although many researchers have reported dynamic formulation in closed form (1), the procedure is very cumbersome, and it is not straightforward to obtain dynamic matrices, $M$, $C$ and $G$ matrices. In this paper we investigate a new general procedure for obtaining closed form dynamic of spherical parallel robots. Several algorithms have been proposed for dynamic formulation of multi body systems that are based on Newton-Euler, Lagrange, and Virtual-Work method. Applying these modeling tools for parallel robots is usually bulky and untractable [10]. The other strategy for dynamic formulation is Gibbs-Appell (GA). This method was first introduced independently by Gibbs (1879) and then by Appell (1899). According to [11] this method provides sufficient means to derive the simplest form of dynamic model. In recent years it has been used to tackle dynamic modeling of complex serial robots such as modular serial robots with large number of joints, [12]. Despite the high potential of this method in dynamic analysis of multi body systems, It has been less addressed in the field of parallel robots.

Dynamic modeling of parallel manipulator has been studied by several researchers through Newton-Euler method, [8], [10]. Newton-Euler method requires force analysis of individual mechanism members, and therefore, having a closed

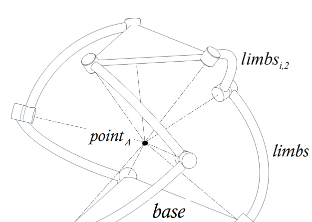


Fig. 1: A schematic of Agile–Wrist

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form dynamic model for the total system needs elimination of all the internal forces, which is a prohibitive task. Lagrange is the other common method for dynamic analysis and in [13] it has been applied for dynamic formulation of various parallel robots. Despite its simple appearance, the nonlinear relation between passive and active joints of the mechanism is a challenge for implementation of Lagrange method for general parallel manipulators. Lagrangian based procedures need symbolic matrix differentiating to obtain the coriolis matrix of dynamic model, which is very difficult to obtain, [14]. Virtual work is the most frequently used method for dynamic modeling of spherical parallel robots, [15], [16]. In [15] the dynamic formulation leads to a recursive model, which is not in closed form, and therefore, not computationally efficient. In this paper a straightforward formulation is proposed, which is an extension of GA method for parallel robots. This method is developed based on GA energy function and robot Jacobian matrix, and is used for dynamic formulation of SPR. In what follows a general description of the GA method is given. This method is then extended to include Jacobian matrices and by this means an explicit form of dynamic matrices $(\boldsymbol{M}, \boldsymbol{C}, \boldsymbol{G})$ is obtained. Since in the proposed method a combination of Gibbs-Appell energy function and Jacobian matrices are used, it can be seen as a medium between Newton-Euler and Lagrange method. Having the benefits of two methods the method derives explicit dynamic model without any symbolic differentiation and recursive computation. As a case study, kinematic analysis and closed form dynamic formulation of agile wrist, which is one of the most well-known 3 degrees of freedom SPR, is obtained. Finally, the verification result is reported for an arbitrary trajectory.

## II. Closed Form Dynamic Formulation

### A. Review of Gibbs–Apell Formulation

In GA method dynamic formulation is derived using Gibbs-Apell function, referred to as acceleration energy function, and is defined as follows, [17]:

$$S = \frac{1}{2} \int (\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{a}})\, dm \tag{2}$$

in witch, $\vec{\boldsymbol{a}}$ denote acceleration of element $dm$. Consider a rigid body with a coordinate system attached to it as shown in Fig. 2. The origin of the body frame, $A$, is a point on the

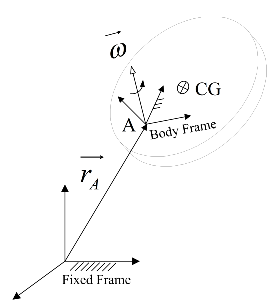


Fig. 2: The schematics of a rigid body

rigid body. The complete form of the GA acceleration energy function for a rigid body is defined as, [17]:

$$S = \frac{1}{2} m \left(\vec{\boldsymbol{a}}_A \cdot \vec{\boldsymbol{a}}_A\right) + \frac{1}{2} \vec{\boldsymbol{\alpha}} \cdot \frac{\partial \vec{\boldsymbol{H}}_A}{\partial t} + \vec{\boldsymbol{\alpha}} \cdot \left(\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{H}}_A\right) + ... \\ m\vec{\boldsymbol{a}}_A \cdot (\vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{\rho}}) + m\vec{\boldsymbol{a}}_A \cdot [\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{\rho}})] + f(\vec{\boldsymbol{v}}, \vec{\boldsymbol{\omega}}) \tag{3}$$

in which the function $f(\vec{\boldsymbol{v}}, \vec{\boldsymbol{\omega}})$ is independant of $\ddot{\boldsymbol{q}}$ . Write the angular momentum as:

$$\vec{\boldsymbol{H}}_A = \boldsymbol{I}_A \vec{\boldsymbol{\omega}} \tag{4}$$

where, $\boldsymbol{I}_A$ is the symmetric inertia tensor and because the time derivative of it in body coordinate is zero, one may write:

$$\frac{\partial \vec{\boldsymbol{H}}_A}{\partial t} = \boldsymbol{I}_A \frac{\partial \vec{\boldsymbol{\omega}}}{\partial t} = \boldsymbol{I}_A \vec{\boldsymbol{\alpha}} \tag{5}$$

Generally, for a multi-body system with $N$ degrees of freedom, GA function, is a function of $N$ generalized coordinates $q_i$ and its derivatives, $\dot{q}_i, \ddot{q}_i$

$$S = S(q_i, \dot{q}_i, \ddot{q}_i) \qquad (i = 1 \dots n) \tag{6}$$

Derive GA energy function with respect to $\ddot{q}_i$ , the governing dynamic equation leads to:

$$\frac{\partial S}{\partial \ddot{q}_i} = \boldsymbol{Q}_i \qquad (i = 1 \dots n) \tag{7}$$

The right side of the equation is generalized forces acting on the body. The matrix form of the equation may be written as:

$$\frac{\partial S}{\partial \ddot{\boldsymbol{q}}} = \boldsymbol{Q} \ \ , \ \ \boldsymbol{q} \ \ \& \ \ \boldsymbol{Q} : [n, 1] \ \ matrix \tag{8}$$

Substituting (4,5) in (8) yields:

$$\frac{\partial S}{\partial \ddot{\boldsymbol{q}}} = m \left(\frac{\partial \vec{\boldsymbol{a}}_A}{\partial \ddot{\boldsymbol{q}}}\right) \cdot \vec{\boldsymbol{a}}_A + \left(\frac{\partial \vec{\boldsymbol{\alpha}}}{\partial \ddot{\boldsymbol{q}}}\right) \cdot (\boldsymbol{I}_A \vec{\boldsymbol{\alpha}}) + ... \\ \left(\frac{\partial \vec{\boldsymbol{\alpha}}}{\partial \ddot{\boldsymbol{q}}}\right) . (\vec{\boldsymbol{\omega}} \times (\boldsymbol{I}_A \vec{\boldsymbol{\omega}})) + m \left(\frac{\partial \vec{\boldsymbol{a}}_A}{\partial \ddot{\boldsymbol{q}}}\right) \cdot (\vec{\boldsymbol{\alpha}} \times \vec{\boldsymbol{\rho}}) + ... \\ m\vec{\boldsymbol{a}}_A \cdot \left(\left(\frac{\partial \vec{\boldsymbol{\alpha}}}{\partial \ddot{\boldsymbol{q}}}\right) \times \vec{\boldsymbol{\rho}}\right) + m \left(\frac{\partial \vec{\boldsymbol{a}}_A}{\partial \ddot{\boldsymbol{q}}}\right) \cdot (\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{\rho}})) \tag{9}$$

The Jacobian matrix for linear and angular velocities of the robot is defined as:

$$\begin{cases} \boldsymbol{V}_A = \boldsymbol{J}_D \dot{\boldsymbol{q}} \\ \boldsymbol{\omega} = \boldsymbol{J}_R \dot{\boldsymbol{q}} \end{cases} \tag{10}$$

In which $\boldsymbol{J}_D$ is the translational Jacobian matrix that maps generalized velocities into linear velocity of the rigid body and $\boldsymbol{J}_R$ denotes that of angular velocity mapping. By differentiating (10) and considering gravity term the following equations for accelerations are obtained:

$$\begin{cases} \boldsymbol{a}_A = \boldsymbol{J}_D \ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_D \dot{\boldsymbol{q}} - \boldsymbol{\sigma} \\ \boldsymbol{\alpha} = \boldsymbol{J}_R \ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_R \dot{\boldsymbol{q}} \end{cases} \tag{11}$$

in which, $\boldsymbol{\sigma}$ denotes the gravity acceleration, writing dynamic equation in this form makes the formulation much simpler and there is no need to include potential energy.

## B. Dynamic Formulation for SPR

Now, for the spherical mechanism only rotational motion exist, therefore, the translational Jacobian $\boldsymbol{J}_D$ and its derivatives are all zero. Thus equation (11) is simplified to:

$$\begin{cases} \boldsymbol{a}_A = -\boldsymbol{\sigma} \\ \boldsymbol{\alpha} = \boldsymbol{J}_R \ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_R \dot{\boldsymbol{q}} \end{cases} \tag{12}$$

Substitute accelerations in the main equation (9):

$$\frac{\partial S}{\partial \ddot{\boldsymbol{q}}} = \left(\frac{\partial \vec{\boldsymbol{\alpha}}}{\partial \ddot{\boldsymbol{q}}}\right) \cdot (\boldsymbol{I}_A \vec{\boldsymbol{\alpha}}) + \left(\frac{\partial \vec{\boldsymbol{\alpha}}}{\partial \ddot{\boldsymbol{q}}}\right) \cdot (\vec{\boldsymbol{\omega}} \times (\boldsymbol{I}_A \vec{\boldsymbol{\omega}})) + \cdots$$
$$m\vec{\boldsymbol{\sigma}} \cdot \left(\vec{\boldsymbol{\rho}} \times \left(\frac{\partial \vec{\boldsymbol{\alpha}}}{\partial \ddot{\boldsymbol{q}}}\right)\right) \tag{13}$$

Apply equation (12), to derive the implicit derivative of angular acceleration as:

$$\frac{\partial \boldsymbol{\alpha}}{\partial \ddot{\boldsymbol{q}}} = \boldsymbol{J}_R \tag{14}$$

Substitute it in (13) and using matrix operator ($^{\times}$) to represent cross product, equation (13) is simplified to:

$$\frac{\partial S}{\partial \ddot{\boldsymbol{q}}} = \boldsymbol{J}_R^T \boldsymbol{I}_A \boldsymbol{\alpha} + \boldsymbol{J}_R^T \boldsymbol{\omega}^{\times} \boldsymbol{I}_A \boldsymbol{\omega} + m\big(\boldsymbol{\rho}^{\times} \boldsymbol{J}_R\big)^T \boldsymbol{\sigma} \tag{15}$$

Substitute equation (12) into (15):

$$\frac{\partial S}{\partial \ddot{\boldsymbol{q}}} = \boldsymbol{J}_R^T \boldsymbol{I}_A \left(\boldsymbol{J}_R \ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}}_R \dot{\boldsymbol{q}}\right) + \cdots$$
$$\boldsymbol{J}_R^T \boldsymbol{\omega}^{\times} \boldsymbol{I}_A \left(\boldsymbol{J}_R \dot{\boldsymbol{q}}\right) + m\big(\boldsymbol{\rho}^{\times} \boldsymbol{J}_R\big)^T \boldsymbol{\sigma} \tag{16}$$

Comparing the last equation with general form of explicit dynamic formulation (1) $\boldsymbol{M}$,$\boldsymbol{C}$,$\boldsymbol{G}$ matrices, for a single rigid-body with pure rotation, is obtained as:

$$\begin{aligned} \boldsymbol{M} &= \boldsymbol{J}_R^T \boldsymbol{I}_A \boldsymbol{J}_R \\ \boldsymbol{C} &= \boldsymbol{J}_R^T \boldsymbol{I}_A \dot{\boldsymbol{J}}_R + \boldsymbol{J}_R^T (\boldsymbol{J}_R \dot{\boldsymbol{q}})^{\times} \boldsymbol{I}_A \boldsymbol{J}_R \\ \boldsymbol{G} &= m\big(\boldsymbol{\rho}^{\times} \boldsymbol{J}_R\big)^T \boldsymbol{\sigma} \end{aligned} \tag{17}$$

For a multi-body system Gibbs-Apell energy function, $S_T$, is the summation of acceleration energy functions of all bodies [17]:

$$S_T = \sum_{i=1}^{N} (S)_i \tag{18}$$

Therefore, dynamic equation of SPR may be written in the form of:

$$\boldsymbol{M}_t \ddot{\boldsymbol{q}} + \boldsymbol{C}_t \dot{\boldsymbol{q}} + \boldsymbol{G}_t = \boldsymbol{Q}_t \tag{19}$$

in which, the dynamic matrices $\boldsymbol{M}, \boldsymbol{C}, \boldsymbol{G}$, may be each represented by $\boldsymbol{X}$ as derived by

$$\boldsymbol{X}_t = \boldsymbol{X}_{ee} + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \boldsymbol{X}_{ij} \tag{20}$$

where,

$$\begin{aligned} &\boldsymbol{M}_{ij} = \boldsymbol{J}_{ij}^T \boldsymbol{I}_{ij} \boldsymbol{J}_{ij} \\ &\boldsymbol{C}_{ij} = \boldsymbol{J}_{ij}^T \boldsymbol{I}_{ij} \dot{\boldsymbol{J}}_{ij} + \boldsymbol{J}_{ij}^T (\boldsymbol{J}_{ij} \dot{\boldsymbol{q}})^{\times} \boldsymbol{I}_{ij} \boldsymbol{J}_{ij} \\ &\boldsymbol{G}_{ij} = \ m_{ij}\big(\boldsymbol{\rho}_{ij}^{\times} \boldsymbol{J}_{ij}\big)^T \boldsymbol{\sigma}_{ij} \ \ , \ \ \boldsymbol{\sigma}_{ij} = \boldsymbol{R}_{ij}^T \vec{\boldsymbol{g}} \end{aligned} \tag{21}$$

Index $i$ is the chain number that enumerates from 1 to $n$ and $j$ indicates the body number in the chain $i$, which enumerates from 1 to $m_i$. Index (ee) is used to represent the end effector. It is very interesting that all dynamic terms are obtained in an explicit form. Now, since $\boldsymbol{I}$ is a positive definite symmetric matrix, the inertia matrix $\boldsymbol{M}$ is obviously determined as a positive definite matrix, too. Matrix $\boldsymbol{C}$ is obtained by using rotational Jacobian and its derivative, and includes centrifugal and coriolis terms. By introducing $\boldsymbol{\sigma}$ within the Jacobian derivatives, gravity term $\boldsymbol{G}$ is derived without any need to introduce and differentiate the potential energy. The systematic way to derive dynamic matrices in an explicit form is the unique feature of the proposed method. The only remaining term in (19) is $\boldsymbol{Q}_t$, that may be simply derived by using the principle of conversation of energy into the overall system.

$$\boldsymbol{Q}_t^T \dot{\boldsymbol{q}} = Power_{ext} \tag{22}$$

By this means, $\boldsymbol{Q}_t$ may be simply derived from the external power injected into the system from the actuators, and other possible external forces/moments.

The method presented in here to obtain the dynamic matrices $\boldsymbol{M}, \boldsymbol{C}, \boldsymbol{G}$ relies on Jacobian matrix and its derivative. To clarify how these terms are obtained in practice, use differential kinematics as follows.

$$\vec{\boldsymbol{\omega}} = \boldsymbol{J} \dot{\boldsymbol{q}} \rightarrow \boldsymbol{J} = \frac{\partial \vec{\boldsymbol{\omega}}}{\partial \dot{\boldsymbol{q}}} \tag{23}$$

Similarly, acceleration analysis may be used to directly derive the derivative of the Jacobian as:

$$\vec{\boldsymbol{\alpha}} = \boldsymbol{J} \ddot{\boldsymbol{q}} + \dot{\boldsymbol{J}} \dot{\boldsymbol{q}} \rightarrow \dot{\boldsymbol{J}} = \frac{\partial \vec{\boldsymbol{\alpha}}}{\partial \dot{\boldsymbol{q}}} \tag{24}$$

in which, partial derivative of the acceleration with respect to $\dot{\boldsymbol{q}}$ is used to derive the Jacobian derivative.

## III. Dynamic Formulation of Agile-Wrist

The kinematic structure of 3DOF spherical parallel robot is shown in Fig. 3. As it is shown in this figure, the robot consists of three identical arms with RRR kinematic structure. The three rotational joints of each arm, and of all arms, intersect at the center point, by which the moving platform of the mechanism has pure rotational motion about it. As it is shown in Fig. 3, the unit vector $\hat{\boldsymbol{u}}_i$ represents the axes of ith input angle, $\hat{\boldsymbol{w}}_i$ and $\hat{\boldsymbol{v}}_i$ denote the axes of motion of intermediate

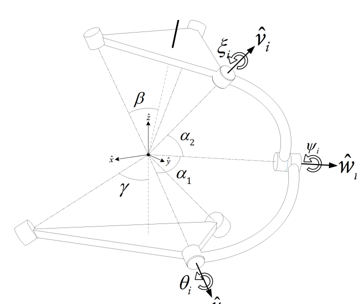

Fig. 3: Kinematic structure of Agile–Wrist

and moving platform joints. All the unit vectors pass through the center of the mechanism, about which, all the mechanism links have pure rotational motion. The angular length of the links are denoted by $\alpha_{i1}, \alpha_{i2}$.

### A. Kinematic Analysis

Using the 321 body-fixed Euler angles for representation of moving platform the rotation matrix may be obtained by:

$$\boldsymbol{R}_{ee} = \boldsymbol{R}_z(q_1)\,\boldsymbol{R}_y(q_2)\,\boldsymbol{R}_x(q_3) \tag{25}$$

By choosing Euler angles as generalized coordinates, angular velocity and acceleration of moving platform is obtained as follow:

$$\boldsymbol{\Omega}_{ee} = \boldsymbol{E}\dot{\boldsymbol{q}} \quad , \; \dot{\boldsymbol{\Omega}}_{ee} = \boldsymbol{E}\ddot{\boldsymbol{q}} + \dot{\boldsymbol{E}}\dot{\boldsymbol{q}} \tag{26}$$

in which:

$$\boldsymbol{E} = \begin{bmatrix} 0 & -sin(q_1) & cos(q_1)cos(q_2) \\ 0 & cos(q_1) & sin(q_1)cos(q_2) \\ 1 & 0 & -sin(q_2) \end{bmatrix}$$

Using successive rotations, rotation matrices of the first and second links are obtained as:

$$\begin{aligned} \boldsymbol{R}_{i1} &= \boldsymbol{R}_z(\lambda_i)\,\boldsymbol{R}_x(\gamma-\pi)\,\boldsymbol{R}_z(\theta_i) \\ \boldsymbol{R}_{i2} &= \boldsymbol{R}_x(\alpha_{1i})\,\boldsymbol{R}_z(\psi_i) \end{aligned} \tag{27}$$

The initial configuration of moving-platform is defined by $\hat{\boldsymbol{v}}_i^*$ as:

$$\hat{\boldsymbol{v}}_i^* = \boldsymbol{R}_z(\eta_i)\,\boldsymbol{R}_x(-\beta)\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-\sin(\eta_i)\sin(\beta)\\ \cos(\eta_i)\sin(\beta)\\ \cos(\beta)\end{bmatrix} \tag{28}$$

that $\gamma, \beta$ are geometrical parameters of robot that have been shown in Fig. 3. The unit vectors $\hat{\boldsymbol{u}}_i$,$\hat{\boldsymbol{v}}_i$ and $\hat{\boldsymbol{w}}_i$ are derived using rotation matrices as follows.

$$\begin{aligned} \hat{\boldsymbol{v}}_i &= \boldsymbol{R}_{ee}\hat{\boldsymbol{v}}_i^* \\ \hat{\boldsymbol{u}}_i &= \boldsymbol{R}_z(\lambda_i)\,\boldsymbol{R}_x(\gamma-\pi)\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-\sin(\lambda_i)\sin(\gamma)\\ \cos(\lambda_i)\sin(\gamma)\\ -\cos(\gamma)\end{bmatrix} \\ \hat{\boldsymbol{w}}_i &= \boldsymbol{R}_z(\lambda_i)\,\boldsymbol{R}_x(\gamma-\pi)\,\boldsymbol{R}_z(\theta_i)\,\boldsymbol{R}_x(\alpha_{1i})\,[0,0,1]^T \end{aligned} \tag{29}$$

Although in the formulations, general structure for the robot is considered, in the simulations the following structural angles are considered for the robot: $\boldsymbol{\eta} = [0,120,240]^\circ$, and $\boldsymbol{\lambda} = [0,120,240]^\circ$.

By dot product the unit vectors $\hat{\boldsymbol{v}}_i$ and $\hat{\boldsymbol{w}}_i$, the angle $\theta_i$ is obtained which is equal to the angular length of mechanism arm $\alpha_{i2}$:

$$\hat{\boldsymbol{v}}_i \cdot \hat{\boldsymbol{w}}_i = \cos(\alpha_{2i}) \tag{30}$$

Write $\hat{\boldsymbol{w}}_i$ in the form:

$$\begin{aligned} \hat{\boldsymbol{w}}_i &= \boldsymbol{R}_z(\lambda_i)\,\boldsymbol{R}_x(\gamma-\pi)\,diag(\sin\alpha_{1i}, -\sin\alpha_{1i}, \cos\alpha_{1i}) \\ &+ [\sin\theta_i, \cos\theta_i, 1]^T \end{aligned} \tag{31}$$

, and define:

$$\boldsymbol{C} = diag(s_{\alpha_{1i}}, -s_{\alpha_{1i}}, c_{\alpha_{1i}})\,\boldsymbol{R}_x^T(\gamma-\pi)\,\boldsymbol{R}^T(\lambda\;)\,\hat{\boldsymbol{v}}\;, \tag{32}$$

equation (30) is simplified to:

$$a.\cos\theta_i + b.\sin\theta_i = c \tag{33}$$

in which, $a = \boldsymbol{C}_{i,2}$, $b = \boldsymbol{C}_{i,1}$, $c = \cos\alpha_2 - \boldsymbol{C}_{i,3}$. Solve the above trigonometric equation for $\theta_i$ and obtain the following two solutions:

$$\theta_i = \begin{cases} atan2(b,a) + atan2\left(\sqrt{a^2+b^2-c^2}, c\right) \\ atan2(b,a) - atan2\left(\sqrt{a^2+b^2-c^2}, c\right) \end{cases} \tag{34}$$

To have a real solution, condition $a^2+b^2-c^2 \geqslant 0$ shall be satisfied. To find, $\psi_i$ the angle between the two normals $\hat{\boldsymbol{n}}_{1i}$ and $\hat{\boldsymbol{n}}_{2i}$ is used:

$$\hat{\boldsymbol{n}}_{1i} = \frac{\hat{\boldsymbol{u}}_i \times \hat{\boldsymbol{w}}_i}{|\sin\alpha_1|}, \;\; \hat{\boldsymbol{n}}_{2i} = \frac{\hat{\boldsymbol{w}}_i \times \hat{\boldsymbol{v}}_i}{|\sin\alpha_2|} \tag{35}$$

Since, $\hat{\boldsymbol{n}}_{1i}, \hat{\boldsymbol{n}}_{2i}$ are perpendicular to $\hat{\boldsymbol{w}}_i$, the angle $\psi_i$ is obtained as follows:

$$\cos\psi_i = \hat{\boldsymbol{n}}_{1i} \cdot \hat{\boldsymbol{n}}_{1i} \tag{36}$$

Apply equations (34) and (36) to solve inverse kinematic of each limb independently.

Now in order to analyze the velocities, the angular velocity of the end effector is obtained as the sum of its successive angular rotations.

$$\dot{\theta}_i\hat{\boldsymbol{u}}_i + \dot{\psi}_i\hat{\boldsymbol{w}}_i + \dot{\xi}\hat{\boldsymbol{v}}_i = \vec{\boldsymbol{\Omega}}_{ee} \tag{37}$$

Dot product equation (37) by $(\hat{\boldsymbol{u}}_i \times \hat{\boldsymbol{v}}_i)$ and $(\hat{\boldsymbol{v}}_i \times \hat{\boldsymbol{w}}_i)$ the unknown $\dot{\theta}_i$ and $\dot{\psi}_i$ are obtained, in explicit form:

$$\dot{\psi}_i = \frac{(\hat{\boldsymbol{u}}_i \times \hat{\boldsymbol{v}}_i)\cdot\vec{\boldsymbol{\Omega}}_{ee}}{(\hat{\boldsymbol{u}}_i \times \hat{\boldsymbol{v}}_i)\cdot\hat{\boldsymbol{w}}_i} \;\; , \;\; \dot{\theta}_i = \frac{(\hat{\boldsymbol{v}}_i \times \hat{\boldsymbol{w}}_i)\cdot\vec{\boldsymbol{\Omega}}_{ee}}{(\hat{\boldsymbol{v}}_i \times \hat{\boldsymbol{w}}_i)\cdot\hat{\boldsymbol{u}}_i} \tag{38}$$

To simplify the equations, define vectors $\vec{\boldsymbol{p}}_{1i}$,$\vec{\boldsymbol{p}}_{2i}$ and scalar parameters $h_i$ as:

$$\begin{aligned} &\vec{\boldsymbol{p}}_{1i} = (\hat{\boldsymbol{v}}_i \times \hat{\boldsymbol{w}}_i) \;\; , \;\; \vec{\boldsymbol{p}}_{2i} = (\hat{\boldsymbol{u}}_i \times \hat{\boldsymbol{v}}_i) \\ &h_i = (\hat{\boldsymbol{v}}_i \times \hat{\boldsymbol{w}}_i)\cdot\hat{\boldsymbol{u}}_i = \vec{\boldsymbol{p}}_{1i}\cdot\hat{\boldsymbol{u}}_i \end{aligned} \tag{39}$$

Using these definitions in (38), to find the joint velocities as:

$$\dot{\theta}_i = \left(\frac{\vec{\boldsymbol{p}}_{1i}}{h_i}\right)\cdot\vec{\boldsymbol{\Omega}}_{ee} \quad \dot{\psi}_i = \left(\frac{\vec{\boldsymbol{p}}_{2i}}{h_i}\right)\cdot\vec{\boldsymbol{\Omega}}_{ee} \tag{40}$$

Differentiate with respect to time, and obtain acceleration relations as follows.

$$\begin{aligned} \ddot{\theta}_i &= \left(\frac{\vec{\boldsymbol{p}}_{1i}}{h_i}\right)\cdot\dot{\vec{\boldsymbol{\Omega}}}_{ee} + \left(\frac{h_i\dot{\vec{\boldsymbol{p}}}_{1i} - \dot{\boldsymbol{h}}_i\vec{\boldsymbol{p}}_{1i}}{h_i^2}\right)\cdot\vec{\boldsymbol{\Omega}}_{ee} \\ \ddot{\psi}_i &= \left(\frac{\vec{\boldsymbol{p}}_{2i}}{h_i}\right)\cdot\dot{\vec{\boldsymbol{\Omega}}}_{ee} + \left(\frac{h_i\dot{\vec{\boldsymbol{p}}}_{2i} - \dot{h}_i\vec{\boldsymbol{p}}_{2i}}{h_i^2}\right)\cdot\vec{\boldsymbol{\Omega}}_{ee} \end{aligned} \tag{41}$$

The angular velocity of the first and second links of each limb are written as follows.

$$\vec{\boldsymbol{\Omega}}_{i1} = \dot{\theta}_i\hat{\boldsymbol{u}}_i \;\; , \;\; \vec{\boldsymbol{\Omega}}_{i2} = \dot{\theta}_i\hat{\boldsymbol{u}}_i + \dot{\psi}_i\hat{\boldsymbol{w}}_i \tag{42}$$

Differentiate with respect to time in order to obtain angular accelerations as follows.

$$\begin{aligned} \dot{\vec{\boldsymbol{\Omega}}}_{i1} &= \ddot{\theta}_i\hat{\boldsymbol{u}}_i \\ \dot{\vec{\boldsymbol{\Omega}}}_{i2} &= \ddot{\theta}_i\hat{\boldsymbol{u}}_i + \ddot{\psi}_i\hat{\boldsymbol{w}}_i + \dot{\psi}_i\left(\dot{\theta}_i\hat{\boldsymbol{u}}_i \times \hat{\boldsymbol{w}}_i\right) \end{aligned} \tag{43}$$

### B. Dynamic Formulation

As it is clear from equations (19,20,21), since in GA formulation all the velocities and accelerations are expressed in body coordinates, in order to perform dynamic formulation only Jacobian matrix and its time derivatives are needed. In order to determine these key elements, write angular velocity relations. ($\boldsymbol{z} = [0,0,1]^T$)

$$\begin{aligned} \vec{\boldsymbol{\omega}}_{i1} &= \boldsymbol{R}_{i1}^T \vec{\boldsymbol{\Omega}}_{i1} = \dot{\theta}_i \hat{\boldsymbol{z}} \\ \vec{\boldsymbol{\omega}}_{i2} &= \boldsymbol{R}_{i1}^T \vec{\boldsymbol{\Omega}}_{i2} = \dot{\theta}_i \left({}^{i1}\boldsymbol{R}_{i2} \hat{\boldsymbol{z}}\right) + \dot{\psi}_i \hat{\boldsymbol{z}} \\ \vec{\boldsymbol{\omega}}_{ee} &= \boldsymbol{R}_{ee}^T \vec{\boldsymbol{\Omega}}_{ee} = \boldsymbol{R}_{ee}^T \boldsymbol{E} \dot{\boldsymbol{q}} \end{aligned} \tag{44}$$

Furthermore, write the acceleration in body coordinates:

$$\begin{aligned} \vec{\boldsymbol{\alpha}}_{i1} &= \boldsymbol{R}_{i1}^T \dot{\vec{\boldsymbol{\Omega}}}_{i1} = \ddot{\theta}_i \hat{\boldsymbol{z}} \\ \vec{\boldsymbol{\alpha}}_{i2} &= \boldsymbol{R}_{i1}^T \dot{\vec{\boldsymbol{\Omega}}}_{i2} = \ddot{\theta}_i \left({}^{i2}\boldsymbol{R}_{i1} \hat{\boldsymbol{z}}\right) + \ddot{\psi}_i \hat{\boldsymbol{z}} + \dot{\psi}_i \dot{\theta}_i \left({}^{i2}\boldsymbol{R}_{i1} \hat{\boldsymbol{z}}\right) \times \hat{\boldsymbol{z}} \\ \vec{\boldsymbol{\alpha}}_{ee} &= \boldsymbol{R}_{ee}^T \dot{\vec{\boldsymbol{\Omega}}}_{ee} = \boldsymbol{R}_{ee}^T \boldsymbol{E} \ddot{\boldsymbol{q}} + \boldsymbol{R}_{ee}^T \dot{\boldsymbol{E}} \dot{\boldsymbol{q}} \end{aligned} \tag{45}$$

Use chain rule in partial derivatives of equation (23,24) as:

$$\begin{aligned} \boldsymbol{J}_{i1} &= \frac{\partial \vec{\boldsymbol{\omega}}_{i1}}{\partial \dot{\boldsymbol{q}}} = \left[\frac{\partial \vec{\boldsymbol{\omega}}_{i1}}{\partial \dot{\theta}_i}\right] \left[\frac{\partial \dot{\theta}_i}{\partial \dot{\boldsymbol{q}}}\right] \\ \boldsymbol{J}_{i2} &= \frac{\partial \vec{\boldsymbol{\omega}}_{i2}}{\partial \dot{\boldsymbol{q}}} = \left[\frac{\partial \vec{\boldsymbol{\omega}}_{i2}}{\partial \dot{\theta}_i}\right] \left[\frac{\partial \dot{\theta}_i}{\partial \dot{\boldsymbol{q}}}\right] + \left[\frac{\partial \vec{\boldsymbol{\omega}}_{i2}}{\partial \dot{\psi}_i}\right] \left[\frac{\partial \dot{\psi}_i}{\partial \dot{\boldsymbol{q}}}\right] \end{aligned} \tag{46}$$

Apply the same method to determine $\dot{\boldsymbol{J}}_{i1}, \dot{\boldsymbol{J}}_{i2}$ from angular accelerations. Since the coriolis acceleration terms of second links need more attention it is given in detail as follows:

$$\begin{aligned} \dot{\boldsymbol{J}}_{i2} = \frac{\partial \vec{\boldsymbol{\alpha}}_{i2}}{\partial \dot{\boldsymbol{q}}} &= \left[\frac{\partial \vec{\boldsymbol{\alpha}}_{i2}}{\partial \ddot{\theta}_i}\right] \left[\frac{\partial \ddot{\theta}_i}{\partial \dot{\boldsymbol{q}}}\right] + \left[\frac{\partial \vec{\boldsymbol{\alpha}}_{i2}}{\partial \ddot{\psi}_i}\right] \left[\frac{\partial \ddot{\psi}_i}{\partial \dot{\boldsymbol{q}}}\right] + \cdots \\ & \frac{1}{2} \left[\frac{\partial \vec{\boldsymbol{\alpha}}_{i2}}{\partial \dot{\theta}_i}\right] \left[\frac{\partial \dot{\theta}_i}{\partial \dot{\boldsymbol{q}}}\right] + \frac{1}{2} \left[\frac{\partial \vec{\boldsymbol{\alpha}}_{i2}}{\partial \dot{\psi}_i}\right] \left[\frac{\partial \dot{\psi}_i}{\partial \dot{\boldsymbol{q}}}\right] \end{aligned} \tag{47}$$

Furthermore, for the end effecter one may write:

$$\boldsymbol{J}_{ee} = \boldsymbol{R}_{ee}^T \boldsymbol{E}, \quad \dot{\boldsymbol{J}}_{ee} = \boldsymbol{R}_{ee}^T \dot{\boldsymbol{E}} \tag{48}$$

in which, for the assigned coordinate system the gravity acceleration vector is given by $\boldsymbol{g} = [0,0,-9.81]^T$.

Assume there is no friction force, and only the gravity and actuators torque are the external forces applied on the robot. According to principle of conservation of energy, one may write

$$\boldsymbol{\tau}^T \dot{\boldsymbol{\theta}} = \boldsymbol{Q}_t^T \dot{\boldsymbol{q}} \tag{49}$$

Use equation (38,39) to define $\boldsymbol{J}_{act}$ as:

$$\dot{\boldsymbol{\theta}} = \boldsymbol{J}_{act} \dot{\boldsymbol{q}} \ , \ \boldsymbol{J}_{act} = \begin{bmatrix} h_1^{-1} \boldsymbol{p}_{11}^T \boldsymbol{E} \\ h_1^{-1} \boldsymbol{p}_{21}^T \boldsymbol{E} \\ h_1^{-1} \boldsymbol{p}_{31}^T \boldsymbol{E} \end{bmatrix} \tag{50}$$

Then, $\boldsymbol{Q}_t$ is obtained by substituting (50) into (49) as:

$$\boldsymbol{Q}_t = \boldsymbol{J}_{act}^{-T} \boldsymbol{\tau}. \tag{51}$$

## IV. Model Verification

For verification purpose consider a symmetric configuration with $\gamma = \cos^{-1}(\sqrt{3}/3)$, and $\beta = \sin^{-1}(\sqrt{3}/3)$. Furthermore, the angular length of limbs are set to $\alpha_{i1} = 80°$, and $\alpha_{i2} = 70°$, respectively, with length density of $1.378 \left[\frac{kg}{m}\right]$. Assume that the end effector has a mass of $1.351[kg]$ with an inertia matrix of, $diag(38.29, 38.29, 4.5)[kg.cm^2]$. Considering these specifications, simulations is performed for a typical trajectory as follows:

$$\begin{cases} q_i = a_i \sin(b_i t) \\ \dot{q}_i = a_i b_i \text{con}\,(b_i t) \\ \ddot{q}_i = -a_i b_i^2 \sin(b_i t) \end{cases} \tag{52}$$

in which, $a_i = [10, 30, 20]^\circ$ , $b_i = [\pi, \pi/2, 3\pi/2]\,(rad/sec)$.

In order to traverse on the prescribed trajectory, we need to generate the corresponding actuator efforts applied to the robot. Such forces are determined in a feedback structure, as shown in Fig. 4. As it is shown in this figure, a high gain PD controller is used with Matlab/SimMechanics model of the robot to develop the robot simulator and to verify the obtained dynamic model. The SimMechanics model consist of three kinematic chains that are linked to end effecter of the mechanism with revolute joint as well as three actuators and required joint sensors.

Both dynamic and kinematic outputs are verified. Since, the proposed dynamic formulation is based on kinematics, and Jacobians, kinematic verification is of more priority.

In order to verify the kinematics solutions, the actual trajectory of simulator output is given to kinematic model and the joint space variables are obtained. The actual trajectory traversed by closed loop robot simulator, and the one obtained from the model are both plotted in Fig. 5, and their difference is shown in top subplot of Fig. 7. As it is seen in these figures, the kinematic outputs are quite identical, with an accuracy of order $10^{-7}$. Furthermore the differential kinematic errors are shown in the second and third subplots of Fig. 7, in which the error in velocity and accelerations are also very small and in order $10^{-6}$, and $10^{-5}$, respectively. Dynamic verification result is shown in Fig. 6. As it is shown in this figure, the difference between the simulator actuator torque to that of the obtained model is of order $10^{-5}$.

## V. Conclusions

Having a closed form dynamic model for parallel robots is of great interest in dynamic analysis, control, calibration and fault detection. Due to complex kinematic structure of parallel

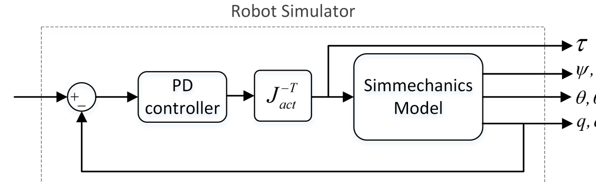


Fig. 4: Block diagram of robot simulator containing robot and controller

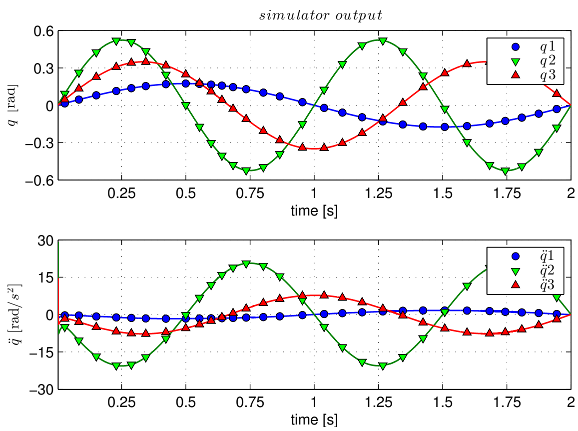


Fig. 5: The simulator and model trajectory

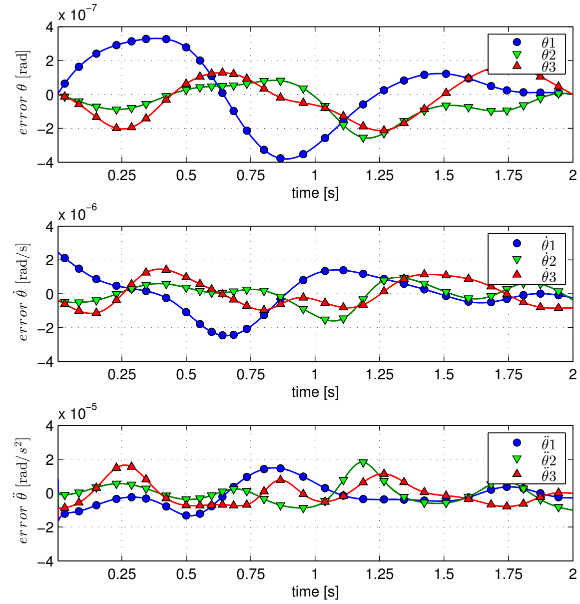


Fig. 7: Results of kinematic verification, using robot simulator: joints variable and its derivative errors.

robot, applying traditional method for dynamic formulation leads to bulky models that are not cost efficient. In this paper a new general and systematic method is proposed to derive closed form dynamic formulation of spherical robots. By rewriting the Gibbs-Appell energy function using Jacobian matrix, dynamic matrices of $\boldsymbol{M}, \boldsymbol{C}, \boldsymbol{G}$ are derived in an explicit form. This method is applied to one of the most celebrated SPRs, namely agile wrist, and the obtained model is verified by simulation. By taking the advantage of this method, dynamic formulation of other SPMs can be achieved in a tractable form. In future works the result of this contribution will be experimentally applied for the spherical eye surgery robot project conducted in ARAS robotic lab.

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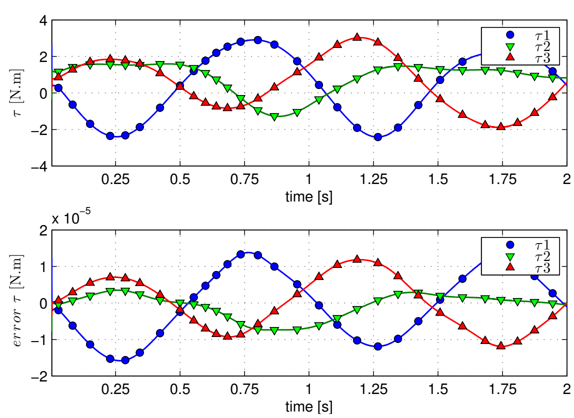


Fig. 6: Dynamic verification: Simulator and dynamic formulation torques, and their error