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Corner stability in nonlinear autonomous systems

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Abstract In most practical applications, studying the asymptotic stability of equilibrium points of a system is of utmost importance. Furthermore, in many cases, the response is restricted to only a sector of the state space. For example, positive systems that are common in chemical processes have nonnegative state variables. For such systems, stability analysis of the system using Lyapunov stability is not advised, since this stability is defined for all the points within a neighborhood of the equilibrium point. In this paper, a new notion of stability, called corner stability, is defined as more suitable for studying asymptotic stability of equilibrium points in such systems. In order to derive sufficient conditions of corner stability, a theorem is stated and proven in this paper, and corner stability of three case studies is analyzed and verified.

Keywords Autonomous systems · Positive system · Stability analysis · Lyapunov stability · Asymptotic stability

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1 Introduction

There are various definitions for stability of a dynamical system. In order to study a history of these definitions, one may refer to reference [1]. We may refer to Poincaré stability (or orbital stability) [2], Zhukovsky stability [3], Lyapunov stability [4], and asymptotic stability [4] as common definitions for the stability of the solution for a dynamical system. Definitions of Poincaré, Zhukovsky, and Lyapunov are equal when analyzing the stability of an equilibrium point. Lyapunov is often chosen to evaluate stability for an equilibrium point among these three definitions. An equilibrium point of a dynamical system is stable in the sense of Lyapunov if all solutions starting at nearby points stay nearby; otherwise, it is unstable. In many practical applications, the stability in the sense of Lyapunov is not sufficient, and convergence to the equilibrium point, not demonstrated by Lyapunov stability, is important; this type of equilibrium point is called asymptotically stable. The basic method to guarantee asymptotic stability is Lyapunov's direct method [4]. In this method, in order to guarantee asymptotic stability of equilibrium point for an autonomous system, the goal is to obtain a positive definite function whose derivative with respect to time is negative definite. In this method, in fact, the problem of asymptotic stability of the equilibrium point is converted to finding a Lyapunov function. Based on Lyapunov theorem, various theorems are generalized with respect to the different approaches such as relaxing the negative definite con-

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dition of the first derivative of a Lyapunov function, using a vector function instead of a scalar function as a Lyapunov function, and applying higher-order derivatives of a Lyapunov function candidate. Briefly, we review the striking results in each of these approaches in the following paragraphs.

LaSalle [5] simplified the conditions of the Lyapunov theorem for asymptotic stability of an autonomous system's equilibrium point by use of invariant sets. In [6], the negative definite condition for the derivative of the function with respect to time is substituted with two conditions, namely derivative of the function should be negative semi-definite with respect to time, and furthermore, there should exist a $T > 0$ where for all $t \geq t_0$, we have $\int_t^{t+T} \dot{V}(x(\tau), \tau)\mathrm{d}\tau \leq -\alpha(\|x(t)\|) < 0$, in which α is a positive monotonic function on $\mathbb{R}^+$ and satisfies $\alpha(0) = 0$. In [7], it is shown that only the second condition given in [6], considering that the function α is in the class of K, can replace the negative definite condition of the derivative function. Furthermore, in reference [8], it is shown that if the second condition in reference [6] is held for a strictly increasing series of time, then it can replace the negative definite condition of the derivative of Lyapunov function for uniform asymptotic stability proof.

Vector Lyapunov function is introduced in [9] which extends the Lyapunov second method for the systems with higher dimensions. In fact, in this method instead of finding a Lyapunov function for the whole system, it is tried to find a decrescent and at least positive semi-definite function in form of $V_i(x, t)$ which for $k_i > 0$ makes the function $V = \sum_{i=1}^{m} k_i V_i$, to become positive definite. Then, V_i is a candidate Lyapunov function of the corresponding subsystem, and function V may be nominated as the Lyapunov function for the whole system [10]. By using the generalized comparison lemma, defined in [11], references [11,12] have used vector Lyapunov function in a different way. They obtain a relation between the solutions of the vector inequality $\dot{V} \leq g(V(x, t))$ and that of $\dot{U} = g(U(x, t))$ with $V, U \in \mathbb{R}^m$. They have shown that if function $g : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is in class of W and for the initial condition we have: $V(x_0, t_0) = U(t_0)$, then we will have: $V_i(x, t) \leq U_i(t)$. Now, if the equilibrium point is asymptotically stable for the system $\dot{U} = g(U(x, t))$, then that equilibrium point will be asymptotically stable for the system in which vector function V is considered for the stability proof. In this method, negative definiteness of $\dot{V}_i(x, t)$ is not required.

Higher-order derivative of a function is used for asymptotic stability proof of a system. For autonomous systems, reference [13] substituted negative definite condition for $\dot{V}(x)$ with condition $a_2(\mathrm{d}\ddot{V}/\mathrm{d}t)(x) + a_1\ddot{V}(x) + \dot{V}(x) < 0$, for all $x \neq 0$, where $a_1, a_2 \geq 0$. In reference [14], the negative definite condition for function $\dot{V}$ is substituted with negative definite condition for $\min\{\dot{V}(x), h\ddot{V}(x)\}$, in which $h > 0$ in some regions around the origin. In reference [13], also it is shown that this condition is held only if $\dot{V}(x)$ is negative definite, and therefore, this condition is useless (origin is considered as the equilibrium point). Reference [15] used the inequality $V^{(m)} \leq g_m(V, \dot{V}, \ldots, V^{(m-1)}, t)$ and compared it with auxiliary system of $u^{(m)} = g_m(u, \dot{u}, \ldots, u^{(m-1)}, t)$ and showed that if the equilibrium point of auxiliary system with the vector field of class W is asymptotic stable, then the equilibrium point of described system with $x(t)$ is asymptotic stable.

In reference [16], which has the combination of above approaches, it is shown that if there exists vector V with decrescent V_i and a positive definite V_1 which satisfies inequality $A_{(m\times m)}\dot{V} \leq [V_2,\ V_3,\ \ldots,\ V_m - \varphi(\|x\|)]^T$, in which $A = [a_{ij}]$ is a lower triangular matrix with $a_{ii} > 0$ and $a_{i>j} \geq 0$, then asymptotic stability of the system is guaranteed (function φ is in class of K).

Recently, the dynamical Lyapunov functions were introduced in the reference [17] which has a different approach from above. Dynamical Lyapunov functions are defined as pair (D_τ, V) in which D_τ is the ordinary differential equation $\dot{\xi} = \tau(x, \xi)$ with $\xi(t) \in \mathbb{R}^n$ and V is a Lyapunov function for an extended system in form of $[\dot{x}\ \dot{\xi}]^T = [f(x)\ \tau(x, \xi)]^T$. Stability proof of the system by using dynamical Lyapunov functions is related to finding the solution, $\xi(x)$, of the equation $\frac{\partial \xi}{\partial x}\ f(x) = \tau(x, \xi(x))$.

In part of references, instead of developing Lyapunov's theorem, efforts have been made to provide methods to obtain a function that satisfies the conditions of Lyapunov's theorem. In reference [18], a Lyapunov function is constructed by backward integration of the composite system trajectory. In reference [19], search for Lyapunov function and its generation, based on the sum of squares (SOS) decomposition, are algorithmically formulated for a class of nonlinear dynamical systems.

All the above-mentioned references are concerned with establishing the stability or asymptotic stability of an equilibrium point. There are also instability theorems for establishing that an equilibrium point is unstable. The most powerful of these theorems is Chetaev's theorem [20]. There are other instability theorems that were proved before Chetaev's theorem, but they are corollaries of the theorem such as Lyapunov's instability theorems. Now, consider that one may want to study asymptotic stability of the equilibrium point of an autonomous system, which its responses are restricted to only a sector of the state space, as an example consider positive system[1] [21,22] that are common in chemical and biological processes [23]. For this purpose, it is sufficient and sometimes necessary (refer to Example 1) that the asymptotic stability definition is restricted to the sector that states are restricted to it. On the other hand, most of the previous works on the stability analysis of such systems are restricted homogeneous [24–26] or linear analysis [27]. Furthermore, for many practical systems, we may encounter physical limitations for existence of solution in a sector of state space. Therefore, it is necessary to develop a sufficient conditions similar to that of the Lyapunov theorem to deal with such systems. This task is accomplished for the first time in this paper.

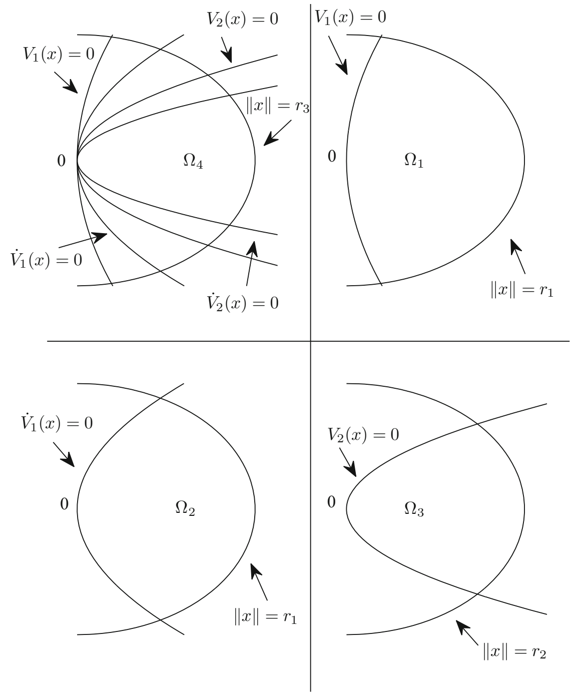

Fig. 1 Geometric representation of sets in the theorem conditions

2 Corner stability

Consider an autonomous system

$$\dot{x} = f(x) \tag{1}$$

where $f : D \to \mathbb{R}^n$ is a locally Lipschitz map from a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Suppose $x_e \in D$ is an equilibrium point of (1); that is, $f(x_e) = 0$. For convenience, we state the definition of corner stability and theorems for the case when the equilibrium point is at the origin of $\mathbb{R}^n$; that is, $x_e = 0$. There is no loss of generality in doing so because any equilibrium point can be shifted to the origin via a change in variables. Suppose $x_e \neq 0$ and consider the change in variables $y = x - x_e$. The derivative of y is given by

$$\dot{y} = \dot{x} = f(x) = f(y + x_e) \overset{\text{def}}{=} g(y), \quad \text{where } g(0) = 0.$$

In the new variable y, the system has equilibrium at the origin. Therefore, without loss of generality, we will always assume that $f(x)$ satisfies $f(0) = 0$ and study the corner stability of the origin $x = 0$.

Definition 1 Equilibrium point on the origin of (1) is called corner stable, if domain $\omega \subset D$ can be found where the origin is a boundary point for this domain and also the set $\Omega = \omega \cup \{0\}$ form a positive invariant set, where each solution $x(t)$ in the set of Ω satisfy[2]

$$\forall \varepsilon > 0,\ \exists \delta > 0 : \|x(t_0)\| < \delta(\varepsilon) \Rightarrow$$
$$\|x(t)\| < \varepsilon,\quad \forall t \geq t_0 \geq 0,\quad \text{and} \quad \lim_{t \longrightarrow \infty} \|x(t)\| = 0. \tag{2}$$

Theorem 1 [20] *Let W be a compact subset of D, $x_0 \in W$, and suppose it is known that every solution of such system (1) with the initial condition $x(t_0) = x_0$ lies entirely in W. Then, there exist a unique solution that is defined for all $t \geq t_0$.*

Theorem 2 *Consider the autonomous system defined by Eq. (1). Now assume that there exist functions $V_1 : D \to \mathbb{R}$ and $V_2 : D \to \mathbb{R}$ with continuous partial first order derivatives, that satisfy the following conditions.*

[1] A positive system is a system in which all of states are non-negative for all $t \geq 0$. Some of practical systems like absolute temperature control, liquid level control in tanks, density control of substances in chemical processes etc. are in positive systems class.

[2] $\|\cdot\|$ Indicates norm.

Then, there exist a region $\Omega \subset D$ in which the corner stability is guaranteed.

Conditions

1. There exist a domain $\Omega_1 \subset D$ where[3]

$$\forall x \in \Omega_1 \Rightarrow V_1(x) > 0, \quad \Omega_1 \subset B(r_1, 0) \subset D, \quad r_1 > 0,$$

   and furthermore, for the boundary of Ω_1, we have[4]

$$\begin{aligned} &\forall x \in \partial\Omega_1 \Rightarrow V_1(x) = 0 \;\vee\; \|x\| = r_1, \\ &0 \in \partial\Omega_1, \quad \partial\Omega_1 \cap B(r_1, 0) \neq \emptyset. \end{aligned} \tag{3}$$

2. There exist a domain $\Omega_2 \subset \Omega_1$ where

$$\forall x \in \Omega_2 \Rightarrow \dot{V}_1(x) < 0,$$

   and furthermore, for the boundary of Ω_2, we have

$$\begin{aligned} &\forall x \in \partial\Omega_2 \Rightarrow \dot{V}_1(x) = 0 \;\vee\; \|x\| = r_1, \\ &0 \in \partial\Omega_2, \quad \partial\Omega_2 \cap B(r_1, 0) = \emptyset, \\ &H = \left\{x \in \partial\Omega_2 \setminus \{0\} : \dot{V}_1(x) = 0\right\}, \\ &[H \cap \Omega_2] \subset \Omega_1. \end{aligned} \tag{4}$$

3. There exist a domain $\Omega_3 \subset \Omega_2$ where

$$\forall x \in \Omega_3 \Rightarrow V_2(x) > 0, \quad \Omega_3 \subset B(r_2, 0), \;\; r_2 > 0,$$

   and furthermore, for the boundary of Ω_3, we have

$$\begin{aligned} &\forall x \in \partial\Omega_3 \Rightarrow V_2(x) = 0 \vee \|x\| = r_2, \quad r_2 < r_1, \\ &0 \in \partial\Omega_3, \quad \partial\Omega_3 \cap B(r_2, 0) \neq \emptyset, \\ &\overline{\Omega}_3 \setminus \{0\} \subset \Omega_2. \end{aligned} \tag{5}$$

4. There exist a domain of $\Omega_4 \subset \Omega_3$ where

$$\begin{aligned} &\forall x \in [\Omega_3 \setminus \Omega_4]^o \Rightarrow \dot{V}_2(x) > 0, \\ &\Omega_4 \subset B(r_3, 0), \quad r_3 > 0, \end{aligned} \tag{6}$$

   and furthermore, for the boundary of Ω_4, we have

$$\begin{aligned} &\forall x \in \partial\Omega_4 \Rightarrow \dot{V}_2 = 0 \;\vee\; \|x\| = r_3, \quad r_3 < r_2, \\ &0 \in \partial\Omega_4, \quad \partial\Omega_4 \cap B(r_3, 0) \neq \emptyset, \\ &[\overline{\Omega}_4 \setminus \{0\}] \subset \Omega_3. \end{aligned} \tag{7}$$

   Refer to Fig. 1

Proof we define domain Ω_3^ε as follows:

$$\Omega_3^\varepsilon = \Omega_3 \cap B(\varepsilon, 0), \quad 0 < \varepsilon < r_3, \tag{8}$$

and define domain π_ρ as follows:

$$\pi_\rho = \left\{x \in \Omega_3^\varepsilon : V_1(x) < \rho\right\}, \quad \rho > 0. \tag{9}$$

since $V_1(x)$ is continuous and domain π_ρ remains non-empty for all $\varepsilon \in (0, r_3)$; therefore, we have

$$0 \in \partial\pi_\rho \tag{10}$$

and ρ satisfies the following inequality

$$0 < \rho < \rho_m \tag{11}$$

where ρ_m calculated as follows:

$$\rho_m = \min_{x \in L} V_1(x), \quad L = \left\{x \in \partial\Omega_3^\varepsilon : \|x\| = \varepsilon\right\}. \tag{12}$$

As L is a compact set and function $V_1(x)$ is continuous on L, therefore, function $V_1(x)$ will have a minimum on L. Also according to Eqs. (4) and (7), we have $L \subset \Omega_1$. Therefore, $V_1(x)$ is positive on L; hence $\rho_m > 0$. Now we may prove that $L \cap \partial\pi_\rho = \emptyset$. For closure π_ρ, we have

$$\begin{aligned} \overline{\pi}_\rho &= \overline{\left[\Omega_3^\varepsilon \cap \{x \in \Omega_1 : 0 < V_1(x) < \rho\}\right]} \\ &\subset \left[\overline{\Omega}_3^\varepsilon \cap \left\{x \in \overline{\Omega}_1 : 0 \leq V_1(x) \leq \rho\right\}\right]. \end{aligned} \tag{13}$$

Therefore, for the right hand of Eq. (13), we have

$$\begin{aligned} &\left[\overline{\Omega}_3^\varepsilon \cap \left\{x \in \overline{\Omega}_1 : 0 \leq V_1(x) \leq \rho\right\}\right] \\ &\quad \subset \left\{x \in \overline{\Omega}_3^\varepsilon : 0 \leq V_1(x) \leq \rho\right\}. \end{aligned}$$

We use a contradiction argument. Suppose $L \cap \partial\pi_\rho \neq \emptyset$. In this case, there exists a point $p_0 \in L$ which is a member of $\partial\pi_\rho$. According to Eq. (12) for the point p_0

[3] $B(r, 0) = \{x \in \mathbb{R}^n : \|x\| < r\}$.

[4] $\partial\Omega_1$, Ω_1^o and $\overline{\Omega}_1$ are boundary, interior and closure of Ω_1, respectively.

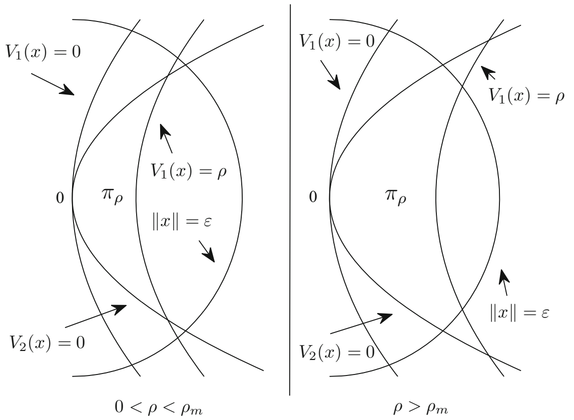

Fig. 2 Geometric representation of set π_ρ

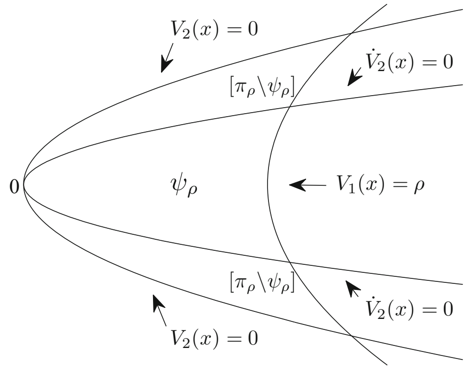

Fig. 3 Geometric representation of set ψ_ρ

on L, we have $V_1(p_0) \geq \rho_m$, and also for $p_0 \in \partial\pi_\rho$, we have $0 \leq V_1(p_0) \leq \rho$ that as $\rho < \rho_m$; therefore, this is a contradiction and $L \cap \partial\pi_\rho = \emptyset$ (Fig. 2). Domain Ω_4^ε is defined as follows

$$\Omega_4^\varepsilon = \Omega_4 \cap B(\varepsilon, 0) \tag{14}$$

and domain of ψ_ρ is defined as follows:

$$\psi_\rho = \left\{x \in \Omega_4^\varepsilon : V_1(x) < \rho\right\}, \quad \rho > 0. \tag{15}$$

Because $V_1(x)$ is continuous and domain ψ_ρ remains nonempty for all $\varepsilon \in (0, r_3)$; therefore, we have

$$0 \in \partial\psi_\rho$$

and furthermore, as $\Omega_4^\varepsilon \subset \Omega_3^\varepsilon$ and according to the Eqs. (9) and (15), it is concluded that $\psi_\rho \subset \pi_\rho$.

Now we may prove that $[\pi_\rho \setminus \psi_\rho] \subset [\Omega_3 \setminus \Omega_4]$. According to Eqs. (8) and (9), we have: $[\pi_\rho \setminus \psi_\rho] \subset \Omega_3$. Therefore, it is sufficient to show that: $[\pi_\rho \setminus \psi_\rho] \cap \Omega_4 = \emptyset$. Assume that $p_0 \in [\pi_\psi \setminus \psi_\rho]$ which means $p_0 \in \pi_\rho$ and $p_0 \notin \psi_\rho$. According to Eq. (15) we have $p_0 \notin \Omega_4$. Therefore, we will have

$$\left[\pi_\rho \setminus \psi_\rho\right] \subset [\Omega_3 \setminus \Omega_4]. \tag{16}$$

For domain π_ρ, we have

$$\forall \in \partial\pi_\rho \Rightarrow V_2(x) = 0 \ \vee \ V_1(x) = \rho, \quad 0 \in \partial\pi_\rho. \tag{17}$$

According to Eqs. (7) and (14), it is concluded that $[\overline{\Omega}_4^\varepsilon \setminus \{0\}] \subset [\overline{\Omega}_4 \setminus \{0\}] \subset \Omega_3$. Hence, we have

$$\forall x \in \partial\psi_\rho \setminus \{0\} \Rightarrow V_1(x) \neq 0. \tag{18}$$

For domain ψ_ρ, we have

$$\forall x \in \partial\psi_\rho \Rightarrow V_1(x) = \rho \ \vee \ \dot{V}_2(x) = 0, \quad 0 \in \partial\psi_\rho, \tag{19}$$

and for the domain π_ρ, it can be written

$$\pi_\rho = \psi_\rho \cup \left[\pi_\rho \setminus \psi_\rho\right].$$

According to Eqs. (17) and (18), boundary of π_ρ defined by $V_2(x) = 0$ is a part of boundary of $[\pi_\rho \setminus \psi_\rho]$ (Fig. 3).

Now consider the solution $x(t)$, in the time interval $[t_0, T)$, is inside π_ρ. According to Eq. (9), we have

$$\forall x \in \pi_\rho \Rightarrow V_1(x) > 0, \quad \dot{V}_1 < 0,$$

and therefore, we have

$$\begin{aligned} V_1(x(t)) &= V_1(x(t_0)) + \int_{t_0}^{t\in[t_0,T)} \dot{V}_1(x(\tau))\mathrm{d}\tau \\ &\Rightarrow V_1(x(t)) \leq V_1(x(t_0)) < \rho. \end{aligned} \tag{20}$$

From Eq. (20), it is concluded that the solution $x(t)$ does not leave the domain π_ρ which its boundaries are specified with $V_1(x) = \rho$. According to Eq. (19), if solution $x(t)$ leaves domain of π_ρ specified by $V_2(x) = 0$, then it will cross region $[\pi_\rho \setminus \psi_\rho]$. According to the Eqs. (6) and (16) for the region $[\pi_\rho \setminus \psi_\rho]$, we have

$$\forall x \in \left[\pi_\rho \setminus \psi_\rho\right]^o \Rightarrow V_2(x) > 0, \quad \dot{V}_2 > 0. \tag{21}$$

Therefore, for the solution $x(t)$ in time interval $[t_0, T)$ to remain inside $[\pi_\rho \setminus \psi_\rho]$, we have

$$\begin{aligned} V_2(x(t)) &= V_2(x(t_0)) + \int_{t_0}^{t\in[t_0,T_1)} \dot{V}_2(x(\tau))d\tau \\ &\Rightarrow V_2(x(t)) \geq V_2(x(t_0)) > 0. \end{aligned} \tag{22}$$

From Eq. (22), it is concluded that the solution $x(t)$ cannot leave domain of π_ρ specified by $V_2(x) = 0$. Therefore, according to Eq. (21), the solution $x(t)$ with the initial condition of $x(t_0) \in \pi_\rho$ remains inside of domain π_ρ for all the times $t \geq t_0$. According to theorem (1) and considering $\pi_\rho \subset \overline{B}(\varepsilon, 0) \subset D$, uniqueness of the solution $x(t)$ is guaranteed. Now according to the fact that by choosing an optional ε in the interval $0 < \varepsilon < r_3$, the set π_ρ is obtained and by choosing δ as follows:

$$\delta_m = \min_{x \in L} \|x\|, \quad L = \left\{x \in \overline{\pi}_\rho : V_1(x) = \rho\right\}$$
$$\Rightarrow \delta < \delta_m,$$

and furthermore, with the notice that: $\pi_\rho \cap B(\delta, 0) \neq \emptyset$ (and because $0 \in \partial\pi_\rho$), the following inequality is proven.

$$\forall \varepsilon > 0,\ \exists \delta > 0 : \|x(t_0)\| < \delta(\varepsilon) \Rightarrow \|x(t)\| < \varepsilon,$$
$$t \geq t_0 \geq 0$$

Now there is only required to prove that

$$\forall x(t_0) \in \pi_\rho \Rightarrow \lim_{t \to \infty} \|x\| = 0.$$

Now we should show that by increasing t to infinity, $V_1(x(t))$ tends to zero because $V_1(x(t))$ is monotonically decreasing on π_ρ and its lower bound is zero. In other words,

$$V_1(x(t)) \to \sigma \geq 0 \quad as \quad t \to \infty, \qquad \sigma < \rho.$$

By using the following contradiction argument, we prove that $\sigma = 0$.

Suppose that $\sigma \neq 0$, then we have $V_1(x(t)) \to \sigma > 0$ which shows that the solution $x(t)$ with initial condition $x(t_0) \in [\pi_\rho \setminus \pi_\sigma]$ for the time interval $[t_0, \infty)$is outside $\pi_\sigma \subset \pi_\rho$. Now calculate α as follows:

$$\alpha = \min_{x \in L}(-\dot{V}_1(x)),$$
$$L = \left\{x \in \overline{\pi}_\rho : \sigma \leq V_1(x) \leq \rho\right\}.$$

According to $[\overline{\pi}_\rho \setminus \{0\}] \subset \Omega_1$ and $0 \notin L$, if there exist an α, then $\alpha > 0$. There exist some values for α because the set L is a compact and function $\dot{V}_1(x)$ is continuous on it. If $x(t)$ in the time interval of $[t_0, \infty)$ stays inside $\pi_\rho \setminus \pi_\sigma$, then we will have

$$V_1(x(t)) = V_1(x(t_0)) + \int_{t_0}^{t} \dot{V}_1(x(\tau))\mathrm{d}\tau$$
$$\leq V_1(x(t_0)) - \alpha(t - t_0), \quad t \geq t_0. \tag{23}$$

According to inequality (23) and $V_1(x(t)) > 0$, we should have

$$t < \frac{V_1(x(t_0))}{\alpha} + t_0. \tag{24}$$

The inequality (24) is in contradiction with the assumption that the solution $x(t)$ in the time interval $[t_0, \infty)$ is inside the $\pi_\rho \setminus \pi_\sigma$. Therefore, $\sigma = 0$ and the proof is complete.

3 Case studies

Example 1 Consider a system described by the following equations.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_1^2 \\ -x_2 \end{bmatrix} \tag{25}$$

According to Chetaev's theorem by using $V = x_1$ for the region $\{x \in \mathbb{R}^2 : x_1 < 0\}$, the instability of origin can be proved.

$$V = x_1 \Rightarrow \dot{V} = -x_1^2 \tag{26}$$

Now, suppose the variable x_1 is the density of a chemical substance, in which, x_1 will always be nonnegative. Therefore, considering instability of the origin based on Chetaev's theorem with using $V = x_1$, it is not true.

The corner stability of origin is studied by using functions $V_1(x)$ and $V_2(x)$ stated in the Eq. (27) containing parameters a and b consequently.

$$V_1 = x_1 + ax_2^2, \qquad V_2 = -x_2^2 - bx_1^4 \tag{27}$$

The derivative of these functions with respect to time is as follows:

$$\dot{V}_1 = -x_1^2 - 2ax_2^2, \qquad \dot{V}_2 = 2x_2^2 + 4bx_1^5.$$

The functions $V_1(x)$ and $V_2(x)$ satisfy the conditions of the theorem by considering $a = -0.4$ and $b = -4$. Therefore, there is $\Omega \subset B(0.4, 0)$ with a boundary

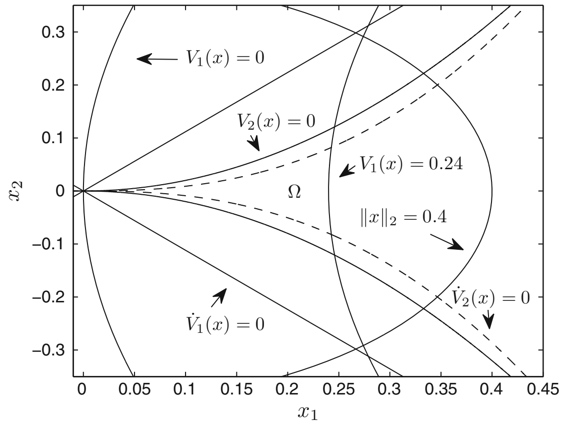

Fig. 4 Geometric representation of sets in the Example 1

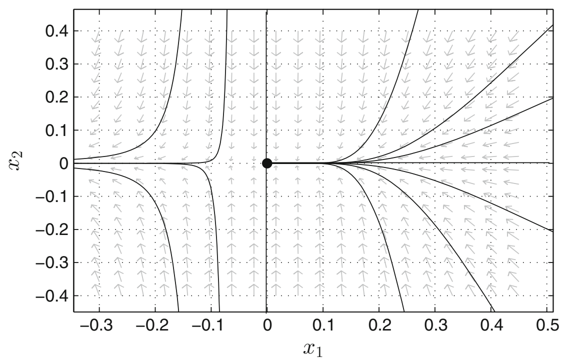

Fig. 5 Phase portrait of a system described by the Eq. (25)

defined by the relation (28) that satisfies corner stability conditions as illustrated in Fig. 4.

$$\forall x \in \partial\Omega : V_1(x) = 0.24 \ \vee \ V_2(x) = 0 \tag{28}$$

Phase portrait of this system is shown in Fig. 5 to illustrate the corner stability of the equilibrium point.

Example 2 Consider a system described by the following equations.

$$\begin{bmatrix} \dot x_1 \\ \dot x_2 \end{bmatrix} = \begin{bmatrix} x_2 - (x_2+1)x_1 \\ 0.2[-x_2 + (x_2+0.5)x_1] \end{bmatrix} \tag{29}$$

This system describes enzyme kinetics. The behavior of this system is studied in [28,29], and estimation of its slow manifold is presented by the following set.

$$M_{\text{slow}} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 = \frac{x_2}{x_2+1} + \frac{0.1x_2}{(x_2+1)^4} \right\}$$

Consider function $V(x)$ and its derivative with respect to time based on the following equations.

$$\begin{aligned} V &= -(x_2 - ax_1)(x_2 - bx_1^3), \qquad a, b = cte, \\ \dot V &= \frac{1}{10}\Big[ax_1^2 - (2+12a)x_1x_2 + (4+10a)x_2^2 \\ &\quad - (4+10a)x_1x_2^2 + 2ax_1^2x_2 + (b+40ab)x_1^4 \\ &\quad - (32b+40ab)x_1^3x_2 + 30bx_1^2x_2^2 \\ &\quad + (2b+40ab)x_1^4x_2 - 30bx_1^3x_2^2\Big] \end{aligned} \tag{30}$$

Now functions $V_1(x)$ and $V_2(x)$ for studying the corner stability of origin are obtained by the equations

$$\begin{aligned} V_1 &= V\ |_{a=20,\ b=0.2}, \\ V_2 &= -\dot V\ |_{a=5,\ b=0.5}\,. \end{aligned} \tag{31}$$

Derivative of function $V_1(x)$ with respect to time can be easily calculated from Eq. (30), and the derivative of function $V_2(x)$ with respect to time is calculated by substituting values of a and b in the Eq. (32).

$$\begin{aligned} \dot V_2 = \frac{-1}{50}\Big[&-(1+16a)x_1^2 + (16+92a)x_1x_2 \\ &- (18+80a)x_2^2 \\ &+ ax_1^3 - (6+54a)x_1^2x_2 + (46+170a)x_1x_2^2 \\ &- (20+50a)x_2^3 - (36b+820ab)x_1^4 \\ &+ (2a+562b+1440ab)x_1^3x_2 \\ &+ (20+50a+300b)x_1x_2^3 \\ &- (8+40a+840ab+600ab)x_1^2x_2^2 \\ &+ (b+20ab)x_1^5 - (124b+1680ab)x_1^4x_2 \\ &+ (1090b+1400ab)x_1^3x_2^2 - 750bx_1^2x_2^3 \\ &+ (40ab+2b)x_1^5x_2 - (100b+800ab)x_1^4x_2^2 \\ &+ 450bx_1^3x_2^3\Big] \end{aligned} \tag{32}$$

Therefore, there is $\Omega \subset B(1,0)$ with a boundary defined by the relation (33) that satisfies corner stability conditions as illustrated in Fig. 6.

$$\forall x \in \partial\Omega \ : \ V_1(x) = 6.11 \ \vee \ V_2(x) = 0 \tag{33}$$

Phase portrait of this system is shown in Fig. 7 to illustrate the corner stability of the equilibrium point.

Example 3 The following example constructed by Vinograd shows that the combination of Lyapunov instability and attractivity can be realized even in an

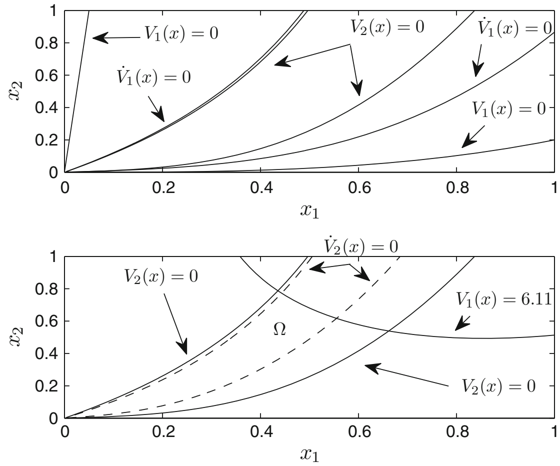

Fig. 6 Geometric representation of the sets in Example 2

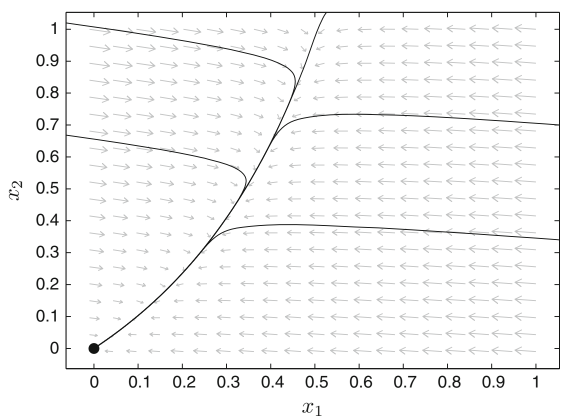

Fig. 7 Phase portrait of a system described by the Eq. (29)

autonomous system of equations of second order [30]. Let

$$\dot{x}_1 = \frac{x_1^2\left(x_2 - x_1 + x_2^5\right)}{\left(x_1^2 + x_2^2\right)\left(1 + \left(x_1^2 + x_2^2\right)^2\right)},$$
$$\dot{x}_2 = \frac{x_2^2\left(x_2 - 2x_1\right)}{\left(x_1^2 + x_2^2\right)\left(1 + \left(x_1^2 + x_2^2\right)^2\right)}. \tag{34}$$

The right sides are defined to be zero for $x_1 = x_2 = 0$. Then, the Lipschitz condition is satisfied. Consider function $V(x)$ and its derivative with respect to time based on the following equations.

$$V = -(x_2 - cx_1)(x_2 - dx_1^2), \qquad c, d = cte,$$
$$\dot{V} = \begin{cases} \frac{1}{(x_1^2+x_2^2)(1+(x_1^2+x_2^2)^2)} w(x), & \text{for} \quad x \neq 0 \\ 0, & \text{for} \quad x = 0 \end{cases}$$

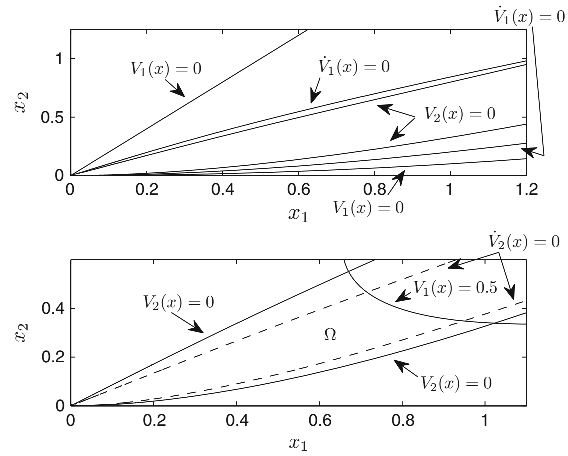

Fig. 8 Geometric representation of the sets in Example 3

$$\begin{aligned} w = & -cx_1^3x_2 + (4+c)x_1x_2^3 - cx_1^2x_2^2 - 2x_2^4 \\ & + 3cdx_1^5 + dx_1^2x_2^3 - (2d + 3cd)x_1^4x_2 \\ & + cx_2^6 + 2dx_1x_2^6 - 3cdx_1^2x_2^5 \end{aligned} \tag{35}$$

Now, functions $V_1(x)$ and $V_2(x)$ for studying the corner stability of origin are obtained by the equations

$$\begin{aligned} V_1 &= V \mid_{c=2,\ d=0.1}, \\ V_2 &= -w \mid_{c=1.5,\ d=0.2}. \end{aligned} \tag{36}$$

Derivative of function $V_1(x)$ with respect to time can be calculated from Eq. (35), and the derivative of function $V_2(x)$ with respect to time is calculated by substituting values of a and b in the Eq. (37).

$$\dot{V}_2 = \begin{cases} \frac{1}{(x_1^2+x_2^2)(1+(x_1^2+x_2^2)^2)} u(x), & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$$
$$\begin{aligned} u = & \ 28x_1x_2^5 + 3cx_1x_2^5 - (20+7c)x_1^2x_2^4 - 4x_1^3x_2^3 \\ & + cx_1^4x_2^2 + 3cx_1^5x_2 - 8x_2^6 - 15cdx_1^7 + 8dx_1^6x_2 \\ & + 27cdx_1^6x_2 - (4d + 6cd)x_1^5x_2^2 \\ & - (4d + 3cd)x_1^4x_2^3 \\ & - 4dx_1^3x_2^4 + 3dx_1^2x_2^5 - 3cx_1^2x_2^6 - 14cx_1x_2^7 \\ & + (4+7c)x_2^8 + 21cdx_1^4x_2^5 + (12cd - 10d)x_1^3x_2^6 \\ & - (22d + 15cd)x_1^2x_2^7 + 14dx_1x_2^8 - 6cdx_1x_2^{10} \\ & + 2dx_2^{11} \end{aligned} \tag{37}$$

Therefore, there is $\Omega \subset B(1.2, 0)$ with a boundary defined by the relation (38) that satisfy corner stability conditions as illustrated in Fig. 8.

$$\forall x \in \partial\Omega \ : \ V_1(x) = 0.4 \ \vee \ V_2(x) = 0 \tag{38}$$

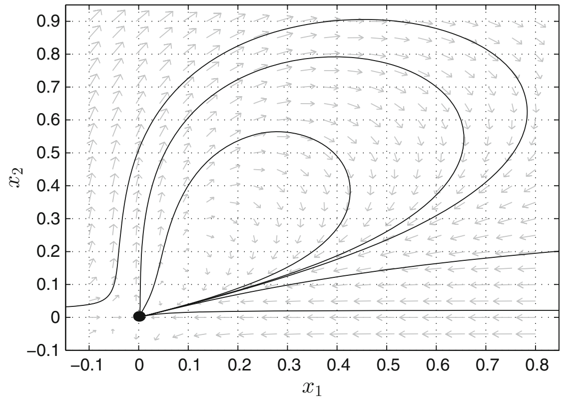

Fig. 9 Phase portrait of a system described by the Eq. (34)

Phase portrait of this system is shown in Fig. 9 to illustrate the corner stability of the equilibrium point.

4 Conclusion

In this study, corner stability has been defined. This kind of stability is a valid replacement for stability assessment of equilibrium point of systems in which their response is limited to a region of state space such as positive systems. Furthermore, a theorem for corner stability is stated and proved in this paper, which provides the sufficient conditions for such stability. Finally, corner stability of equilibrium point of three case studies has been analyzed, and the results are verified by their phase portraits.

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