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Comment on: "Centers of quasi-homogeneous polynomial planar systems" [Nonlinear Anal. RWA 13 (2012) 419]

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ABSTRACT

We describe a counter-example which shows that (**2**) of theorem (**11**) in Algaba et al. (2012) is not correct. This part of the theorem, pinpoints whether the origin of quasi-homogeneous system (**15**) in Algaba et al. (2012) is a center or not. It is shown in this note that the given necessary and sufficient conditions of theorem (**11**), in Algaba et al. (2012) are not complete.

1. Introduction

Let us first give some definitions about quasi-homogeneous vector fields [1].

- Let $\mathbf{t} = (t_1, t_2)$ be non-null, with t_1 and t_2 non-negative integer coprime numbers and $t_1 \leq t_2$. A function $p : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is quasi-homogeneous of type $\mathbf{t}$ and degree k if $p(\epsilon^{t_1} x, \epsilon^{t_2} y) = \epsilon^k p(x, y)$. The vector space of quasi-homogeneous of type $\mathbf{t}$ and degree k will be denoted by $\mathscr{P}_k^{\mathbf{t}}$.
- A polynomial vector field $\mathbf{F} = (P, Q)^T$ is quasi-homogeneous of type $\mathbf{t} = (t_1, t_2)$ and degree r (we denote it $\deg(\mathbf{F}) = r$) if $P \in \mathscr{P}_{r+t_1}^{\mathbf{t}}$ and $Q \in \mathscr{P}_{r+t_2}^{\mathbf{t}}$. We will denote by $\mathcal{Q}_r^{\mathbf{t}}$ the vector space of the quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree r.
- A singular point is called center-focus type, also called monodromic, when the orbits rotate around the singular point.

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The system (**15**) in [1], depending on the type $\mathbf{t}$, has been expressed as follows:

$$\mathbf{t} = (1,2), \qquad \begin{cases} \dot{x} = a_{20}x^2 + a_{01}y, \\ \dot{y} = b_{30}x^3 + b_{11}xy. \end{cases} \tag{\textbf{15}}$$

Furthermore, theorem (**11**) in [1] states the following:

1. The origin of (**15**) is monodromic if and only if $(b_{11} - 2a_{20})^2 + 8b_{30}a_{01} < 0$.
2. If the origin of system (**15**) is monodromic, then it is a center if and only if $2a_{20} = -b_{11}$.
3. If the origin of system (**15**) is a center, then system (**15**) is reversible and analytically integrable.

2. Counter—Example

Consider the system

$$\mathbf{t} = (1,2), \qquad \begin{cases} \dot{x} = x^2 - y, \\ \dot{y} = x^3. \end{cases} \tag{1}$$

System (**15**) is reduced to (1), setting $a_{20} = 1$, $a_{01} = -1$, $b_{30} = 1$, and $b_{11} = 0$. Furthermore, notice that $(b_{11} - 2a_{20})^2 + 8b_{30}a_{01} < 0 \Rightarrow -4 < 0$ and $2a_{20} \neq -b_{11} \Rightarrow 2 \neq 0$. Hence, based on theorem (**11**), the origin of system (1) shall be a monodromic point and is not a center. However, based on its analytical solution, we illustrate that the origin of system (1) is a center.

Changing variable $z = x^2y^{-1}$, equations of system (1) can be reduced to an equation with separable variables, as follows:

$$\frac{1}{y}dy = \frac{-z}{z^2 - 2z + 2}dz. \tag{2}$$

Integrating relation (2) gives:

$$F = \ln(y) + \frac{1}{2}\ln(z^2 - 2z + 2) - \tan^{-1}(1 - z). \tag{3}$$

Eq. (3) can be rewritten as follows:

$$W = (x^4 - 2x^2y + 2y^2)e^{2\tan^{-1}\left(\frac{x^2}{y} - 1\right)}. \tag{4}$$

According to the relation (4), the function V is defined as follows:

$$V = \begin{cases} (x^4 - 2x^2y + 2y^2)e^{2\tan^{-1}\left(\frac{x^2}{y} - 1\right)}, & y > 0, \\ x^4e^{\pi}, & y = 0, \\ (x^4 - 2x^2y + 2y^2)e^{\left[2\tan^{-1}\left(\frac{x^2}{y} - 1\right) + 2\pi\right]}, & y < 0. \end{cases} \tag{5}$$

Now, it can be proven that the function V is continuous on $\mathbb{R}^2$. For the domains $L^+ = \{(x,y) : y > 0\}$ and $L^- = \{(x,y) : y < 0\}$, it is obvious that V is continuous and also for $(x_c, 0) \in L = \{(x,y) : y = 0\}$ and $x_c \neq 0$, we have

$$\begin{cases} \lim\limits_{\substack{(x,y)\longrightarrow(x_c,0) \\ (x,y)\in L^+}} V = x_c^4 \lim\limits_{\substack{(x,y)\longrightarrow(x_c,0) \\ (x,y)\in L^+}} e^{2\tan^{-1}\left(\frac{x^2}{y} - 1\right)} = x_c^4e^{\pi}, \\ \lim\limits_{\substack{(x,y)\longrightarrow(x_c,0) \\ (x,y)\in L}} V = x_c^4e^{\pi}, \\ \lim\limits_{\substack{(x,y)\longrightarrow(x_c,0) \\ (x,y)\in L^-}} V = x_c^4 \lim\limits_{\substack{(x,y)\longrightarrow(x_c,0) \\ (x,y)\in L^-}} e^{\left[2\tan^{-1}\left(\frac{x^2}{y} - 1\right) + 2\pi\right]} = x_c^4e^{\pi}. \end{cases} \tag{6}$$

According to the relations (5) and (6) hitherto, the function V is continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$. By the squeeze theorem [2] and $0 < e^{-\pi} < e^{2\tan^{-1}\left(\frac{x^2}{y}-1\right)} < e^{\pi}$, we conclude that

$$\lim_{\substack{(x,y)\longrightarrow(0,0)\\(x,y)\in L^+}} V = 0, \qquad \lim_{\substack{(x,y)\longrightarrow(0,0)\\(x,y)\in L}} V = 0, \qquad \lim_{\substack{(x,y)\longrightarrow(0,0)\\(x,y)\in L^-}} V = 0. \tag{7}$$

Therefore, V is continuous at $(0,0)$.

We now prove that the partial derivatives of V are continuous on $\mathbb{R}^2$. For $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ at $(x,y) \in [\mathbb{R}^2 \setminus L]$, we have

$$\begin{aligned}
\frac{\partial V}{\partial x} &= \begin{cases} 4x^3 e^{2\tan^{-1}\left(\frac{x^2}{y}-1\right)} & \text{if } (x,y)\in L^+,\\ 4x^3 e^{\left[2\tan^{-1}\left(\frac{x^2}{y}-1\right)+2\pi\right]} & \text{if } (x,y)\in L^-,\end{cases}\\
\frac{\partial V}{\partial y} &= \begin{cases} (4y-4x^2) e^{2\tan^{-1}\left(\frac{x^2}{y}-1\right)} & \text{if } (x,y)\in L^+,\\ (4y-4x^2) e^{\left[2\tan^{-1}\left(\frac{x^2}{y}-1\right)+2\pi\right]} & \text{if } (x,y)\in L^-.\end{cases}
\end{aligned} \tag{8}$$

Hence, the partial derivatives of V are continuous on L^+ and L^-.

Furthermore, the partial derivatives of V at any point $(x,y) \in L$ are:

$$\begin{aligned}
&\frac{\partial V}{\partial x} = 4x^3 e^{\pi},\\
&\frac{\partial V}{\partial y} = \lim_{h\longrightarrow 0} \frac{V(x,h)-V(x,0)}{h} \Longrightarrow\\
&\begin{cases} \lim\limits_{h\longrightarrow 0^+} \dfrac{(x^4-2x^2h+2h^2)e^{2\tan^{-1}\left(\frac{x^2}{h}-1\right)} - x^4e^{\pi}}{h} = -4x^2e^{\pi},\\ \lim\limits_{h\longrightarrow 0^-} \dfrac{(x^4-2x^2h+2h^2)e^{\left[2\tan^{-1}\left(\frac{x^2}{h}-1\right)+2\pi\right]} - x^4e^{\pi}}{h} = -4x^2e^{\pi}.\end{cases}
\end{aligned} \tag{9}$$

Considering relations (8) and (9), it can be shown that $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ are continuous on $\mathbb{R}^2$. Therefore, referring to the following theorem [2]:

> If the partial derivatives $\frac{\partial V}{\partial x}$ and $\frac{\partial V}{\partial y}$ exist near (x,y) and are continuous at (x,y), then V is differentiable at (x,y).

V is differentiable on $\mathbb{R}^2$.

Regarding to the equations of system (1), it can be easily shown that the derivative of V with respect to time is zero on $\mathbb{R}^2$. Therefore, the contour which is determined by $V(x,y) = c$ with constant c, is a union of orbits of system (1). On the other hand, through following inequality,

$$V \geq e^{-\pi}(x^4 - 2x^2y + 2y^2) = e^{-\pi}[(x^2-y)^2 + y^2] \tag{10}$$

it is proved that V is coercive [3]. Hence, the origin of system (1) is a center. The phase portrait of this system is shown in Fig. 1, which verifies the proof of argument given in this note.

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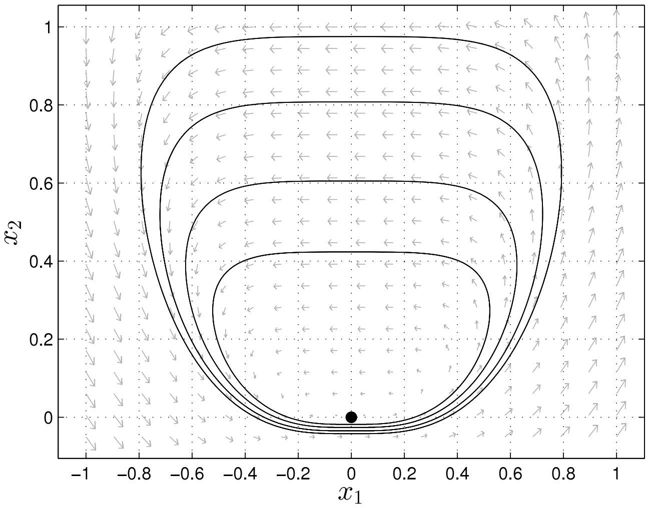

Fig. 1. Phase portrait of system (1).

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