# کنترل غیرخطی

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## Solution1:

Take $V(x) = -\frac{1}{6}x_1^6 + \frac{1}{4}x_2^4$

$$\dot{V}(x) = x_1^6 + x_2^6 - x_1^5 x_2^6 + x_2^3 x_1^6$$

Near the origin

$$\left|-x_1^5 x_2^6 + x_2^3 x_1^6\right| \le \frac{1}{2}(x_1^6 + x_2^6)$$

Hence

$$\dot{V}(x) \ge \frac{1}{2}(x_1^6 + x_2^6)$$

Which shows that $\dot{V}(x)$ is positive definite. Application of Chetaev's theorem shows that the origin is unstable.

## Solution2:

To ensure $\dot{V}(x) < 0 \rightarrow g^T(x)f(x) < 0$

$$\dot{V} = g1x_2 - ((x_1 + x_2) + h(x_1 + x_2))g2$$

$$g = \begin{bmatrix} g1 \\ g2 \end{bmatrix} = \begin{bmatrix} a(x)x_1 + b(x)x_2 \\ c(x)x_1 + d(x)x_2 \end{bmatrix} \quad \text{and} \quad \frac{\partial g1}{\partial x_2} = \frac{\partial g2}{\partial x_1}$$

$$\text{Then}: \quad (I)\ \ b + \frac{\partial a}{\partial x_2}x_1 + \frac{\partial b}{\partial x_2}x_2 = c + \frac{\partial d}{\partial x_1}x_2 + \frac{\partial c}{\partial x_1}x_1$$

$$\dot{V} = ax_1x_2 + bx_2{}^2 - cx_1{}^2 - cx_1x_2 - dx_1x_2 - dx_2{}^2 - (cx_1 + dx_2)h(x_1 + x_2)$$

To cancel the cross-product term we choose

$$ax_1x_2 - cx_1x_2 - dx_1x_2 = 0 \quad \text{and} \quad c = d = cte$$

$$\dot{V} = bx_2{}^2 - cx_1{}^2 - cx_2{}^2 - c(x_1 + x_2)h(x_1 + x_2)$$

$$\text{and}\ \ b = c = cte \quad \text{and} \quad c > 0$$

$\dot{V} = -cx_1{}^2 - c(x_1 + x_2)h(x_1 + x_2) \quad \text{for all } x_1, x_2 \ \dot{V} < 0$

$ax_1x_2 - cx_1x_2 - dx_1x_2 = 0 \ \text{ then } \ a - 2c = 0 \ \text{ then : } a = 2c$

$(I) \ : \quad b + 0 * x_1 + 0 * x_2 = c + 0 * x_2 + 0 * x_1 \ \text{ then } \ b = c$

$$g = \begin{bmatrix} 2x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} = c\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$V = c\int_0^{x_1} g1(y_1, 0)dx_1 + c\int_0^{x_2} g2(x_1, x_2)dx_2$$

$V = c(x_1{}^2 + x_1x_2 + 0.5x_2{}^2) = c((x_1 + 0.5x_2)^2 + 0.25x_2{}^2) \quad \text{and } c > 0$

$\text{for all} \quad x_1, x_2 \neq (0,0) \ : \ V(x_1, x_2) > 0$

The eq. point is globally asymptotically stable

## Solution3:

$$V = \frac{\beta x_2{}^2}{\beta(\alpha + \beta\cos(x_1))^2} + \frac{\gamma}{\beta(\alpha + \beta\cos(x_1))^2} - \frac{\gamma}{\beta(\alpha + \beta)^2} + 0.5\rho(2x_2 + x_3)^2$$

$$\frac{\gamma}{\beta(\alpha + \beta\cos(x_1))^2} - \frac{\gamma}{\beta(\alpha + \beta)^2} > 0 \ \text{ for } \forall x_1$$

For all $\quad x_1, x_2 \neq (0,0) \ : \ V(x_1, x_2, x_3) > 0$

$$\dot{V} = \frac{2x_2\dot{x}_2}{\beta(\alpha + \beta\cos(x_1))^2} + \frac{2\sin(x_1)(\gamma + \beta x_2{}^2)\dot{x}_1}{\beta(\alpha + \beta\cos(x_1))^3} + \rho(2\dot{x}_2 + \dot{x}_3)(2x_2 + x_3)$$

$2\dot{x}_2 + \dot{x}_3 = -u_d \quad \& \quad \dot{x}_1 = x_2$

$$\dot{V} = \frac{-2x_2u_d}{\beta(\alpha + \beta\cos(x_1))^2} - \frac{2x_2\sin(x_1)(\gamma + \beta x_2{}^2)}{\beta(\alpha + \beta\cos(x_1))^3} + \frac{2x_2\sin(x_1)(\gamma + \beta x_2{}^2)}{\beta(\alpha + \beta\cos(x_1))^3} - \rho u_d(2x_2 + x_3)$$

$$\dot{V} = -\left(\frac{2x_2}{\beta(\alpha + \beta\cos(x_1))^2} + \rho(2x_2 + x_3)\right)u_d$$

$\dot{V} = -u_d{}^2 \ \text{ Then } \quad \dot{V} \leq 0 \quad \dot{V} \ \text{ is n.s.d.} \quad \dot{x}_1 = 0 \ \text{ Then : } \ x_2 = 0$

According to Lassalle:

$$\dot{x}_2, \dot{x}_3 = 0 \ \text{ then}: \begin{cases} -\dfrac{\gamma \sin(x_1)}{\alpha + \beta \cos(x_1)} - \rho x_3 = 0 \\ \dfrac{2\gamma \sin(x_1)}{\alpha + \beta \cos(x_1)} + 2\rho x_3 = 0 \end{cases}$$

$x_3, x_1 = 0$ , Then : (0,0) is asymptotically stable.

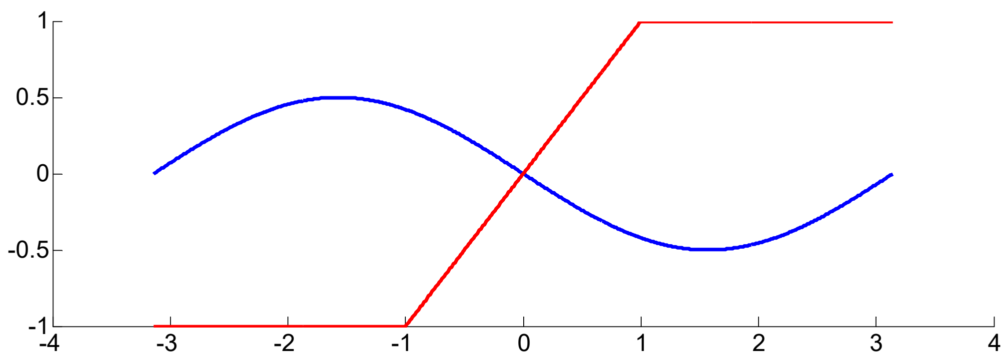
## Solution4:

**1.**

$\dot{x}_1 = 0 \ \ then \ \ x_2 = 0$

$\dot{x}_2 = 0 \ \ then \ \ -\sin(x_1) - 2sat(x_1) = 0$

$sat(x) = -0.5\sin(x)$




$x_1 = 0$

Therefore, the only answer for the equilibrium point of the system is $(0,0)$ .

**2.**

$$\dot{x}_1 = \frac{\partial f_1}{\partial x_1}|_{(x1,x2)=(0,0)} + \frac{\partial f_1}{\partial x_2}|_{(x1,x2)=(0,0)}$$

$$\dot{x}_2 = \frac{\partial f_2}{\partial x_1}|_{(x1,x2)=(0,0)} + \frac{\partial f_2}{\partial x_2}|_{(x1,x2)=(0,0)}$$

$$\begin{cases} \dot{x}_1 \\ \dot{x}_2 \end{cases} = \begin{matrix} x_2 \\ -3x_1 - 3x_2 \end{matrix}$$

$$A = \frac{\partial f}{\partial x}|x = x^* = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

$$A = \frac{\partial f}{\partial x}|x^* = (0,0) = \begin{bmatrix} 0 & 1 \\ -3 & -3 \end{bmatrix} \implies \lambda_1 = -1.5 + 0.86i \ \& \ \lambda_2 = -1.5 - 0.86i \implies stable\ Focus$$

In the linear approximation we have $(0,0)$ $is\ \ stable\ Focus\ \ then\ \ it\ is$ asymptotically stable.

**3.**

$\sigma = x_1 + x_2$

$\dot{x}_1 = x_2$

$\dot{x}_2 = -x_2 - sin(x_1) - 2sat(x_1 + x_2)$

$for\ \ \sigma > 1\ \ \ sat(\sigma) = 1$

$0.5\sigma^2 = 0.5(x_1 + x_2)^2\ :\ \ \ \sigma\dot{\sigma} = (x_1 + x_2)(\dot{x_1} + \dot{x_2}) = -\sigma\big(\sin(x_1) + 2sat(\sigma)\big)$

$for\ \ \sigma > 1\ \ \ \sigma\dot{\sigma} = -\sigma(\sin(x_1) + 2)\ \ \ and\ \ \ \sin(x_1) + 2 \leq 3 \rightarrow \sigma\dot{\sigma} \geq -\sigma$

$for\ \ \sigma < -1\ \ \sigma\dot{\sigma} = -\sigma(\sin(x_1) - 2)\ \ \ and\ \ \ \sin(x_1) - 2 \leq -1$

$\sigma\dot{\sigma} \leq \sigma$

$\sigma\dot{\sigma} \leq \sigma\ \ and\ \ \ \sigma\dot{\sigma} > -\sigma\ \ \ then:\ \ \ \sigma\dot{\sigma} \leq -|\sigma|$

**4.**

$V = {x_1}^2 + 0.5{x_2}^2 + 1 - cos(x_1)$

$\dot{V} = 2x_1x_2 - x_2\big(x_2 + sin(x_1) + 2sat(x_1 + x_2)\big) + x_2\,sin(x_1)$

$|\sigma| < 1\ \ sat(\sigma) = \sigma\ \ and\ \ \ \dot{V} = \ -3{x_2}^2$

According to Lassalle and paragraph 1:

$\dot{x}_1 = 0\ \ Then:\ \ \ x_2 = 0\ \ \ and\ \ \ \dot{x}_2 = 0\ \ Then: -sin(x_1) - 2sat(x_1) = 0$

$Then:\ \ \ x_1 = 0\ \ \ \ \ and\ \ \ (0,0)\ \ is\ \ a.s.\ \ \ and\ \ invariant\ set$

$for\ :\ |\sigma| < 1\ \ \ \ \sigma^2 = (x_1 + x_2)^2$

$2\sigma\dot{\sigma} = 2(x_1 + x_2)(\dot{x_1} + \dot{x_2}) = -2\sigma\big(sin(x_1) + 2sat(\sigma)\big)$

$for\ \ \sigma = 1 \quad \dot{V} = -2\,sin(x_1) - 4 < 0 \quad \forall\, x_1$

$for\ \ \sigma = -1 \quad \dot{V} = 2\,sin(x_1) - 4 < 0 \quad \forall\, x_1$

$then\ \ \sigma = 1 \qquad V = {x_1}^2 + 0.5{x_2}^2 + 1 - cos(x_1)$

$V = 0.5(x_1 + x_2)^2 + 1.5{x_1}^2 - x_1(x_2 + x_1) + 1 - cos(x_1) \qquad \sigma = 1$

$V = 1.5 - cos(x_1) + 1.5{x_1}^2 - x_1 and\ V = 0.5{x_1}^2 + (x_1 - 0.5)^2 + 1.25 = c > 0$

$for\ \ \sigma = -1 \quad V = 0.5{x_1}^2 + (x_1 + 0.5)^2 + 1.25 = c > 0$

$then\ \ : \quad M_c = \{x\,\epsilon\,R^2 | V(x) \le c\} \cap \{x\,\epsilon\,R^2|\ |\sigma| \le 1\}\ \ and\ \ c > 0$

For all $x_1$ values, the proposed Lyapunov function has a positive value and a negative derivative, so Region of Attraction is obtained in the above form.

**5.** According to the paragraphs before that $\sigma\dot{\sigma} \le -|\sigma|$ will converge from region $|\sigma| \ge 1$ to region $|\sigma| \le 1$ and Also, by entering the trajectory in this area, it will definitely converge to the origin. Also, according to the first part, $(0,0)$ is the only equilibrium point of the system. So equilibrium point of origin is globally asymptotically stable.

## Solution5:

1. From the figure it can be seen that

$$\dot{x}_1 = -g(e) + 2x_2 - x_1$$
$$\dot{x}_2 = +g(e) - x_2$$
$$e = -x_1$$

and the system is given by:

$$\dot{x}_1 = x_1^3 + 2x_2 - x_1$$
$$\dot{x}_2 = -x_1^3 - x_2$$

2. Clearly the function $V(x)$ is positive definite and radially unbounded. The derivative of $V(x)$ along the trajectories of the system is given by:

$$\dot{V}(x) = \frac{1}{2}\,\dot{x}^T P x + \frac{1}{2} x^T P \dot{x}$$

$$\dot{V}(x) = -x_1^2 - x_2^2 - 2x_1^3 x_2$$

$$\dot{V}(x) = -x_1^2 - x_2^2 - 2x^T \begin{bmatrix} 0 & \frac{1}{2}x_1^2 \\ \frac{1}{2}x_1^2 & 0 \end{bmatrix} x$$

$$\dot{V}(x) = -x^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x - x^T \begin{bmatrix} 0 & x_1^2 \\ x_1^2 & 0 \end{bmatrix} x$$

$$\dot{V}(x) = -x^T \begin{bmatrix} 1 & x_1^2 \\ x_1^2 & 1 \end{bmatrix} x$$

$$\dot{V}(x) = -x^T Q(x) x$$

Where positive definiteness of $Q(x)$ implies that the origin is asymptotically stable. In order for $Q(x)$ to be positive definite, is is required that all its leading principal minors are positive. This imposes the requirements:

$$1 > 0$$

$$1 - x_1^4 > 0$$

Taking $D = \{x \in R^2 | \, |x_1| < 1\}$ and applying Theorem 4.1 (Page 114, Khalil), shows that the origin is asymptotically stable.