Exercise 1: Consider the system

$\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1 - c(t)x_2$

where the function $c(t)$ is continuous differentiable and satisfies

$k_1 \le c(t) \le k_2$ and $|\dot{c}(t)| \le k_3 \; \forall t \ge 0$

and k_i are constants and $k_1 > 0$. show that the origin is uniformly stable and that $x_2 \to 0$ as $t \to \infty$.

Exercise 2: Consider the following nonlinear system

$\dot{x}_1 = -\phi(t)x_1 + a\phi(t)x_2$

$\dot{x}_2 = b\phi(t)x_1 - ab\phi(t)x_2 - c\psi(t)x_2^3$

where a, b and c and are positive constants and $\phi(t)$, $\psi(t)$ are positive definite, continuous and bounded functions, meanwhile

$$\phi(t) \ge \phi_0 > 0 \quad \text{and} \quad \psi(t) \ge \psi_0 > 0 \quad , \quad t \ge 0$$

show that the origin is globally uniformly asymptotically stable. Is the origin also exponentially stable?

Exercise 3: The purpose of this exercise is to evaluate the exponentially stability of the system in presence of bounded disturbance. Consider the following nonlinear system

$$\dot{x} = f(t,x) + g(t,x) \qquad (*)$$

$$x \in R^n \quad , \quad \|g(t,x)\| \le \mu\|x\|$$

Suppose that $f(t,0) = g(t,0) = 0$ and the origin of nominal system ($\dot{x} = f(t,x)$) is exponentially stable with the Lyapunov function $V(x) = x^T x$. Show that the origin of $(*)$ is exponentially stable if μ is small enough.

Exercise 4: Show the following systems are ISS and find the ultimate bound.

(a) $\dot{x} = -x^3 + xu.$ (b) $\dot{x} = -x + u^3$ $(c)\ \begin{aligned}\dot{x} &= -x^3 + xy \\ \dot{y} &= -y + u^3\end{aligned}$