Solution1:

Consider the system

$\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1 - c(t)x_2$

A Lyapunov function candidate is taken as

$$\mathrm{V(x)} = \frac{1}{2}\left(\mathrm{x}_1^2 + \mathrm{x}_2^2\right)$$

The time derivative of V (x) along the trajectories of the system is found as

$$\dot{\mathrm{V}}(\mathrm{x}) = \mathrm{x}_1(x_2) + \mathrm{x}_2(-x_1 - c(t)x_2) = -c(t)x_2 \le -k_1x_2^2$$

and it can be seen that $\dot{\mathrm{V}}(\mathrm{x})$ is negative semidefinite. By Theorem 4.8 we conclude that the origin is uniformly stable ($\mathrm{V}(\mathrm{x})$ is positive definite and decresent). In order to prove that $x2 \to 0$ as $t \to \infty$ we apply Barbalat's lemma. Since $\dot{\mathrm{V}}(\mathrm{x}) = -k_1x_2^2$, where $c(t)$ is some bounded value greater than zero, $\dot{\mathrm{V}}(\mathrm{x}) = 0 \leftrightarrow x_2 = 0$. Following the notation of Lemma 8.2, let $\phi(t) = \dot{\mathrm{V}}(t)$. $\dot{\mathrm{V}}(t)$ is uniformly continuous in t if $\ddot{\mathrm{V}}(t)$ is bounded:

$$\ddot{\mathrm{V}}(t) = -\dot{\mathrm{c}}(\mathrm{t})\mathrm{x}_2^2 - 2\mathrm{c}(\mathrm{t})\mathrm{x}_2\dot{\mathrm{x}}_2$$

$$\ddot{\mathrm{V}}(t) = -\dot{\mathrm{c}}(\mathrm{t})\mathrm{x}_2^2 - 2\mathrm{c}(\mathrm{t})\mathrm{x}_2(-x_1 - c(t)x_2)$$

it can be recognized that $V(t) \le V(t0)$, which implies that x_1 and x_2 are bounded. Since x_1 and x_2 are bounded and it is given that $c(t)$ and $\dot{\mathrm{c}}(\mathrm{t})$ are bounded, it follows that $\ddot{\mathrm{V}}(t)$ is bounded. The bound on $\ddot{\mathrm{V}}(t)$ guarantees that $\dot{\mathrm{V}}(\mathrm{t})$ is uniformly continuous. In order to conclude by Barbalat's lemma we also need to prove that $\lim_{t\to\infty}\int_0^t \dot{\mathrm{V}}(\tau)\, d\tau$exists and is finite. This is proven according to:

$$\lim_{t\to\infty}\int_0^t \dot{\mathrm{V}}(\tau)\, d\tau = \lim_{t\to\infty}(V(t) - V(0))$$

where we know that l$\lim_{t\to\infty} V(t) = V(\infty)$ is a finite number since $V(t) \ge 0 \forall t$ and $\dot{\mathrm{V}}(\mathrm{t}) \le 0 \forall t$.

Solution2:

$V = \frac{1}{2}(bx_1^2 + ax_2^2)$ and

$w_1(x) = \frac{1}{4}(bx_1^2 + ax_2^2) \leq \frac{1}{2}(bx_1^2 + ax_2^2) \leq (bx_1^2 + ax_2^2) = w_2(x)$

$\dot{V} = -b\emptyset(t)x_1^2 + ab\emptyset(t)x_1x_2 + ab\emptyset(t)x_1x_2 - a^2b\emptyset(t)x_2^2 - ac\,\psi(t)x_2^4$

$\dot{V} = -b\emptyset(t)x_1^2 + 2ab\emptyset(t)x_1x_2 - a^2b\emptyset(t)x_2^2 - ac\,\psi(t)x_2^4$

$\dot{V} \leq -b\emptyset(t)(x_1^2 - 2ax_1x_2 - a^2x_2^2) \leq -b\emptyset_0(x_1^2 - 2ax_1x_2 - a^2x_2^2)$

$\dot{V} \leq -b\emptyset_0(x_1 - ax_2)^2 \quad then\ \dot{V}$ is Negative semi – definite

$V = \frac{1}{2}(bx_1^2 + ax_2^2)$ is lower bounded

$\dot{V} = -b\emptyset(t)x_1^2 + 2ab\emptyset(t)x_1x_2 - a^2b\emptyset(t)x_2^2 - ac\psi(t)x_2^4$

then $\dot{V}$ is uniformly continus.

Having the above three conditions and considering the barbalat's Lemma , we conclude that the system will converges to the origin when $\dot{V} = 0$

Linearization:

$\begin{cases}\dot{x_1} \\ \dot{x_2}\end{cases} = \begin{matrix}-\emptyset(t)x_1 + a\emptyset(t)x_2 \\ b\emptyset(t)x_1 - ab\emptyset(t)x_2\end{matrix} \quad$ then : $V = \frac{1}{2}(bx^2{}_1 + ax^2{}_2)$ and

$w_1(x) = \frac{1}{4}(bx^2{}_1 + ax^2{}_2) \leq \frac{1}{2}(bx^2{}_1 + ax^2{}_2) \leq (bx^2{}_1 + ax^2{}_2) = w_2(x)$

$\dot{V} = -b\emptyset(t)x^2{}_1 + ab\emptyset(t)x_1x_2 + ab\emptyset(t)x_1x_2 - a^2b\emptyset(t)x^2{}_2$

$\dot{V} = -b\emptyset(t)(x_1 - ax_2)^2 \leq -b\emptyset_0(x_1 - ax_2)^2 = -b\emptyset_0 x^T \begin{bmatrix}1 & -a \\ -a & a^2\end{bmatrix} x$

$then:\ \dot{V} \leq -b\emptyset_0 x^T \begin{bmatrix}1 & -a \\ -a & a^2\end{bmatrix} x$

$and\ \ A = \begin{bmatrix}1 & -a \\ -a & a^2\end{bmatrix}$ is positive semi – definite $and\ x = 0$ is not exponentially stable

Solution3:

Discussion on the system $f(t,x)$:

Because $\dot{y} = f(t,y)$ *is e.s. then* $V = y^T y$ *and* $0.5\|y\|^2 < y^T y < 2\|y\|^2$

$$\dot{V} = \dot{y}^T y + y^T \dot{y}$$

We have $\dot{V} = f^T y + y^T f \le -y^T P y$ and P is positive definite.

Discussion on the overall system:

Spouse that: $V = x^T x$ *and* $0.5\|x\|^2 < x^T x < 2\|x\|^2$

$$then: \quad \dot{V} = \dot{x}^T x + x^T \dot{x} = (f^T + g^T)x + x^T(f+g)$$

Then $\dot{V} = f^T x + x^T f + g^T x + x^T g \le -x^T P x + g^T x + x^T g$

$$Then: \quad \dot{V} \le -x^T P x + \|g^T x + x^T g\| \le -x^T P x + 2\|x^T g\|$$

$$\dot{V} \le -x^T P x + 2\mu\|x^T x\| \ then: \dot{V} \le (-\|P\| + 2\mu)\|x\|^2$$

$$\text{then} \quad x = 0 \ \ is \ e.s. \ if \ \ Q = \|P\| - 2\mu > 0 \quad \mu < \|P\|/2$$

where P is matrix induced norm. $\|P\| = \sup_{\|x\|=1} \|Px\|$

Solution4:

(a) $V = \frac{1}{2}x^2, \dot{V} = -x^4 + x^2 u$

$$= -(1-\theta)x^4 - \theta x^4 + x^2 u \le -(1-\theta)x^4 \ for \frac{\sqrt{|u|}}{\theta} \le \|x\|, \ 0 < \theta < 1$$

$$\gamma(r) = \rho(r) = \sqrt{r}/\theta .$$

(b) $V = \frac{1}{2}x^2, \dot{V} = -x^2 + xu^3$

$$-(1-\theta)x^2 - \theta x^2 + xu^3 \le -(1-\theta')x^2 \ for \ |x| \le \frac{|u|^3}{\theta'}, 0 < \theta' < 1$$

$$\gamma'(r) = \rho'(r) = \frac{r^3}{\theta'}.$$

(c) For the cascade system:

The first subsystem is ISS in y and the second subsystem is ISS in u so the whole system is ISS in u with ultimate bound $\gamma_t = r^{\frac{3}{2}}/\theta\sqrt{\theta'}$