Question 1)

$$^{A}R_{B}=\begin{bmatrix}0.099 & -0.3696 & 0.9239\\ 0.9839 & -0.1124 & -0.1464\\ 0.1487 & 0.9235 & 0.3536\end{bmatrix}$$

$$\begin{cases}u^{T}.u=1\\ v^{T}.v=10021\neq 1\\ w^{T}.w=1.0001\neq 1\end{cases}\quad,\quad\begin{cases}x^{T}.x=1\\ y^{T}.y=1.0021\neq 1\\ z^{T}.z=1\end{cases}\quad\Rightarrow\quad R_{22}\text{ and }R_{23}\text{ are incorrect.}$$

$$(-0.3696)^{2}+R_{22}^{2}+0.9235^{2}=1\Rightarrow R_{22}=\pm 0.1027$$
$$(-0.9239)^{2}+R_{23}^{2}+0.3536^{2}=1\Rightarrow R_{23}=\pm 0.1462$$

Thus, there is four solution, although the relation of $\det(R)=1$ is satisfied only in one of them:

$$R_{22}=-0.1027\quad,\quad R_{23}=-0.1462$$

Question 2)

$$\begin{aligned}P_{z}&=P_{1}\cos\theta+(\hat{s}\times P_{1})\sin\theta+(P_{1}.\hat{s})(1-\cos\theta)\hat{s}\\&=\left(\cos\theta+\hat{s}\times\sin\theta+(1-\cos\theta)\hat{s}\,\hat{s}^{T}\right)P_{1}\\&=\left(\hat{s}\,\hat{s}^{T}+\cos\theta\left(1-\hat{s}\,\hat{s}^{T}\right)+\hat{s}_{\times}\sin\theta\right)P_{1}\end{aligned}$$

$$R=\left(\hat{s}\,\hat{s}^{T}+\cos\theta\left(1-\hat{s}\,\hat{s}^{T}\right)+\hat{s}_{\times}\sin\theta\right)[1]$$

We can write the McLorn exponent of the rotation matrix as follows:

$$R(\theta)=R(0)+R'(0)+\cdots+\frac{R^{n}(0)}{n!}\theta^{n}+\cdots$$

According to cross product:

$$\hat{s}_{\times}^{2n+1}=(-1)^{n}\,\hat{s}_{\times}$$
$$\hat{s}_{\times}^{2n}=(-1)^{n}\left(1-\hat{s}_{\times}\hat{s}_{\times}^{T}\right)[2]$$

Derivative of [1] and using [2]:

$$R^{n}(0)=\hat{s}_{\times}^{n}$$

$$R(\theta) = R(0) + R'(0) + \cdots + \frac{R^n(0)}{n!}\theta^n + \cdots = 1 + \hat{s}_\times \theta + \cdots + \frac{\hat{s}_\times^n}{n!}\theta^n + \cdots = e^{\hat{s}_\times \theta}$$

Question 3)

Based on information given, ${}^{A}P_1 = \begin{bmatrix} 10 \\ 10 \\ 4 \end{bmatrix}$, ${}^{B}P_1 = \begin{bmatrix} 8 \\ 20 \\ -2 \end{bmatrix}$

a) In this part, it is desired to find ${}^{A}P_{O_B}$. It can be obtained using ${}^{A}P_1 = {}^{A}T_B \, {}^{B}P_1$, since:

$$ {}^{A}T_B = \begin{bmatrix} {}^{A}R_B & {}^{A}P_{O_B} \\ 0_{1\times3} & 1 \end{bmatrix} = \begin{bmatrix} \cos 30° & 0 & \sin 30° & {}^{A}x_{o_B} \\ 0 & 1 & 0 & {}^{A}y_{o_B} \\ -\sin 30° & 0 & \cos 30° & {}^{A}z_{o_B} \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Hence,

$$ {}^{A}P_1 = {}^{A}T_B \, {}^{B}P_1 \quad \Rightarrow \quad \begin{bmatrix} 10 \\ 10 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.866 & 0 & 0.5 & {}^{A}x_{o_B} \\ 0 & 1 & 0 & {}^{A}y_{o_B} \\ -0.5 & 0 & 0.866 & {}^{A}z_{o_B} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 20 \\ -2 \\ 1 \end{bmatrix} $$

$$ {}^{A}P_{O_B} = \begin{bmatrix} 4.0718 \\ -10 \\ 9.732 \end{bmatrix} $$

b) In this part it is desired to find ${}^{B}P_{O_A}$. Based on ${}^{A}P_{O_B}$ obtained in part (a), ${}^{B}P_{O_A}$ can be obtained using ${}^{B}P_{O_A} = -{}^{A}R_B^T \, {}^{A}P_{O_B}$:

$$ {}^{B}P_{O_A} = -{}^{A}R_B^T \, {}^{A}P_{O_B} = -\begin{bmatrix} 0.866 & 0 & -0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 0.866 \end{bmatrix} \begin{bmatrix} 4.0718 \\ -10 \\ 9.732 \end{bmatrix} = \begin{bmatrix} 1.3397 \\ 10 \\ -10.4641 \end{bmatrix} $$

$$ {}^{B}P_{O_A} = \begin{bmatrix} 1.3397 \\ 10 \\ -10.4641 \end{bmatrix} $$

Question 4)

By assigning intermediate coordinate sytems *B*, *C*, *D* and *E* as figure(1), consecutive transformation can be used to find ${}^{F}A$ (coordinates of point A in *F* coordinate system):

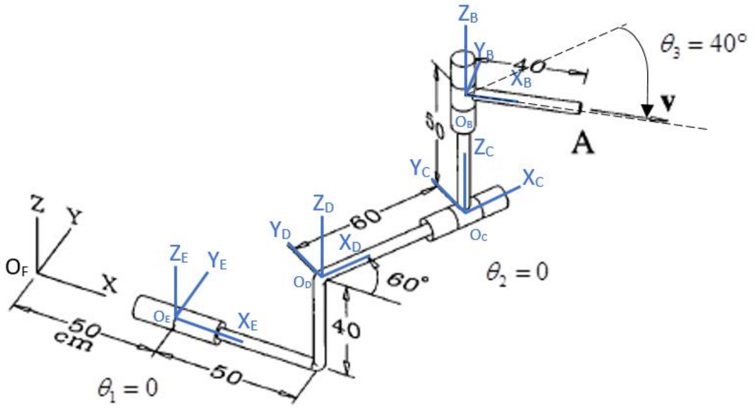

Figure (1)

$$^{F}A = {}^{F}T_{B}\, {}^{B}A$$

In which,

$$^{F}T_{B} = {}^{F}T_{E}\, {}^{E}T_{D}\, {}^{D}T_{C}\, {}^{C}T_{B}$$

E coordinate system can be obtained by transforming fixed *F* coordinate system 50cm along its X-axis without any rotation, thus $^{F}R_{E} = I_{3\times3}$, and homogenous transformation becomes:

$$^{F}T_{E} = \begin{bmatrix} ^{F}R_{E} & ^{F}P_{O_E} \\ 0_{1\times3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & ^{F}x_{O_E} \\ 0 & \cos\theta_1 & -\sin\theta_1 & ^{F}y_{O_E} \\ 0 & \sin\theta_1 & \cos\theta_1 & ^{F}z_{O_E} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 50 \\ 0 & \cos 0^\circ & -\sin 0^\circ & 0 \\ 0 & \sin 0^\circ & \cos 0^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 50 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

D coordinate system can be obtained by transforming *E* coordinate system 50cm along its X-axis, 40 cm along its Z-axis and rotating it 60 degrees about Z-axis based on right hand rule, thus $^{E}R_{D} = R_{Z}(60^\circ)$, and homogenous transformation becomes:

$$^{E}T_{D} = \begin{bmatrix} ^{E}R_{D} & ^{E}P_{O_D} \\ 0_{1\times3} & 1 \end{bmatrix} = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ & 0 & ^{E}x_{O_D} \\ \sin 60^\circ & \cos 60^\circ & 0 & ^{E}y_{O_D} \\ 0 & 0 & 1 & ^{E}z_{O_D} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.866 & 0 & 50 \\ 0.866 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 40 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since second joint has no rotation ($\theta_2 = 0$), *C* coordinate system can be obtained merely by transforming *D* coordinate system 60cm along its X-axis:

$$^{D}T_{C} = \begin{bmatrix} ^{D}R_{C} & ^{D}P_{O_C} \\ 0_{1\times3} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & ^{D}x_{O_C} \\ 0 & \cos\theta_2 & -\sin\theta_2 & ^{D}y_{O_C} \\ 0 & \sin\theta_2 & \cos\theta_2 & ^{D}z_{O_C} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 60 \\ 0 & \cos 0^\circ & -\sin 0^\circ & 0 \\ 0 & \sin 0^\circ & \cos 0^\circ & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 60 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The *B* coordinate system assigned to last joint can be obtained by transforming *C* coordinate system 50 cm along its Z-axis and rotating -40 degrees about Z-axis based on right hand rule,though ${}^{C}R_B = R_Z(-40°)$, and homogenous transformation becomes:

$$ {}^{C}T_B = \begin{bmatrix} {}^{C}R_B & {}^{C}P_{O_B} \\ 0_{1\times3} & 1 \end{bmatrix} = \begin{bmatrix} \cos(-\theta_3) & -\sin(-\theta_3) & 0 & {}^{C}x_{O_B} \\ \sin(-\theta_3) & \cos(-\theta_3) & 0 & {}^{C}y_{O_B} \\ 0 & 0 & 1 & {}^{C}z_{O_B} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(-40°) & -\sin(-40°) & 0 & 0 \\ \sin(-40°) & \cos(-40°) & 0 & 0 \\ 0 & 0 & 1 & 50 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.766 & 0.6428 & 0 & 0 \\ -0.6428 & 0.766 & 0 & 0 \\ 0 & 0 & 1 & 50 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Hence,

$$ {}^{F}T_B = {}^{F}T_E\,{}^{E}T_D\,{}^{D}T_C\,{}^{C}T_B = \begin{bmatrix} 1 & 0 & 0 & 50 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & -0.866 & 0 & 50 \\ 0.866 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 40 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 60 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.766 & 0.6428 & 0 & 0 \\ -0.6428 & 0.766 & 0 & 0 \\ 0 & 0 & 1 & 50 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{F}T_B = \begin{bmatrix} {}^{F}R_B & {}^{F}P_{O_B} \\ 0_{1\times3} & 1 \end{bmatrix} = \begin{bmatrix} 0.9397 & -0.342 & 0 & 130 \\ 0.342 & 0.9397 & 0 & 51.96 \\ 0 & 0 & 1 & 90 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{F}A = {}^{F}T_B\,{}^{B}A \quad \Rightarrow \quad \begin{bmatrix} {}^{F}x_A \\ {}^{F}y_A \\ {}^{F}z_A \\ 1 \end{bmatrix} = \begin{bmatrix} 0.9397 & -0.342 & 0 & 167.5866 \\ 0.342 & 0.9397 & 0 & 65.6382 \\ 0 & 0 & 1 & 90 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 40 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 167.5866 \\ 65.6382 \\ 90 \\ 1 \end{bmatrix} $$

$$ {}^{F}A = \begin{bmatrix} 167.5866 \\ 65.6382 \\ 90 \end{bmatrix} $$

Unit vector ***v*** is equivalent to X-axis of B coordinate system, thus ***v*** can be expressed as $v = {}^{F}\hat{x}_B$.

Rotation matrix ${}^{F}R_B$,that obtained above by ${}^{F}T_B = \begin{bmatrix} {}^{F}R_B & {}^{F}P_{O_B} \\ 0_{1\times3} & 1 \end{bmatrix}$, can be used to find ${}^{F}\hat{x}_B$,

$$ {}^{F}R_B = \begin{bmatrix} {}^{F}\hat{x}_B & {}^{F}\hat{y}_B & {}^{F}\hat{z}_B \end{bmatrix} = \begin{bmatrix} 0.9397 & -0.342 & 0 \\ 0.342 & 0.9397 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ v = {}^{F}\hat{x}_B = \begin{bmatrix} 0.9397 \\ 0.342 \\ 0 \end{bmatrix} $$

Question 5)

a)

$$\theta = \arccos\left(\frac{\text{trace (R)} - 1}{2}\right) = \arccos(0.500) = 60$$

$$S_x = \frac{0.7904 + 0.6237}{2^*0.866} = 0.8164$$

$$S_y = \frac{0.5202 + 0.1868}{2^*0.866} = 0.4082$$

$$S_z = \frac{0.5202 + 0.1868}{2^*0.866} = 0.4082$$

b)

$$\epsilon_1 = S_x \sin(\theta/2) = 0.4082$$
$$\epsilon_2 = S_y \sin(\theta/2) = 0.2041$$
$$\epsilon_3 = S_z \sin(\theta/2) = 0.2041$$
$$\epsilon_4 = \cos(\theta/2) = 0.866$$

c)

$$R = \begin{bmatrix} \cos z1 & -\sin z1 & 0 \\ \sin z1 & \cos z1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos y & 0 & \sin y \\ 0 & 1 & 0 \\ -\sin y & 0 & \cos y \end{bmatrix} \begin{bmatrix} \cos z2 & -\sin z2 & 0 \\ \sin z2 & \cos z2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$z1 = \text{atan } 2(r_{23}, r_{13}) = -0.8756\text{rad} = -50.2\text{deg}$$

$$y = \text{atan } 2\left(\sqrt{r_{23}^2 + r_{13}^2}, r_{33}\right) = 0.9480\text{rad} = 54.3\text{deg}$$

$$z2 = \text{atan } 2(r_{32}, -r_{31}) = 1.3387\text{rad} = 76.7\text{deg}$$