

### **Robotics: Mechanics & Control**



### **Chapter 2: Motion Description**

In this chapter we first briefly review the robot components, and then elaborate on spatial motion description. For this means, position and orientation representation by rotation matrix, screw axis, quaternions and Euler angles are introduces. General motion of a rigid body is then represented by Chasles's theorem, homogeneous transformation and screw axis representation.

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# Welcome

### **To Your Prospect Skills**

On Robotics :

**Mechanics and Control** 



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### About ARAS

**ARAS** Research group originated in 1997 and is proud of its 22+ years of brilliant background, and its contributions to the advancement of academic education and research in the field of Dynamical System Analysis and Control in the robotics application. **ARAS** are well represented by the industrial engineers, researchers, and scientific figures graduated from this group, and numerous industrial and R&D projects being conducted in this group. The main asset of our research group is its human resources devoted all their time and effort to the advancement of science and technology. One of our main objectives is to use these potentials to extend our educational and industrial collaborations at both national and international levels. In order to accomplish that, our mission is to enhance the breadth and enrich the quality of our education and research in a dynamic environment.

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## Contents

#### **Robot Components**

Links and joints, primary joints, compound joints, robot kinematic structures, Some serial robot structures.

#### **Spatial Motion Description**

Coordinate systems, position and orientation representation, rotation matrix, rotation matrix properties, screw axis, unit quaternion, Euler angles.

#### **General Motion of a Rigid Body**

Chasles' Theorem, rotation plus orientation, screw axis representation.

#### **Homogeneous Transformation**

Definition, consecutive transformation, inverse transformation, finite and infinitesimal angle rotation.

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- Robot Components:
  - $\checkmark$  Links and Joints
    - Rigid Links
    - Flexible Links





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- Robot Components:
  - ✓ Primary Joints

### Revolute (R)



### Prismatic (P)



### 1 Rotary DoF

### 1 Translational DoF

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- Robot Components:
  - ✓ Compound Joints
    - Universal (U)



### Spherical (S)







2R DoFs

PR: TR DoF

3R DoFs

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- Robot Kinematic Structure
  - ✓ Crank and Piston (Planar)

<u>RRRP</u>





RRPP

### Single Kinematic Loop

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- Robot Kinematic Structure
  - ✓ Chain of Kinematic Loops (Planar)



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- Robot Kinematic Structure
  - ✓ Single Kinematic Chain (Spatial)



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- Robot Kinematic Structure
  - ✓ Open Kinematic Chain (Spatial)



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- Robot Kinematic Structure
  - ✓ Different Serial Robots (Planar)





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- Robot Kinematic Structure
  - ✓ Different Serial Robots (Planar)



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- **Robot Kinematic Structure** 
  - ✓ Different Serial Robots (Spatial)



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SHOULDER JOINT 2

WRIST ROLL JOINT 4 532°

ELBOW JOINT 3 270

WRIST BEND JOINT 5 200°

> WRIST SWIVEL JOINT 6 600°



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Coordinate systems, position and orientation representation, rotation matrix, rotation matrix properties, screw axis, unit quaternion, Euler angles.

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Chasles' Theorem, rotation plus orientation, screw axis representation.

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Definition, consecutive transformation, inverse transformation, finite and infinitesimal angle rotation.

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- Coordinate Systems
  - ✓ Cartesian coordinate
    - Reference Frame {A}
       With origin @ O<sub>A</sub>
       and {x̂, ŷ, ẑ} unit direction in space.
    - Moving Frame {B}
       With origin @ O<sub>B</sub>
       and {û, v̂, ŵ} unit direction in space.
  - ✓ Cylindrical or Spherical



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- Spatial Motion Description
  - $\checkmark$  Position of a point in rigid body
    - With reference to frame {*A*}

 ${}^{A}\boldsymbol{P} = \begin{bmatrix} \boldsymbol{P}_{x} \\ \boldsymbol{P}_{y} \\ \boldsymbol{P}_{z} \end{bmatrix}$ 

- Superscript *A* denote the reference frame
- Index {x, y, z} denote the Cartesian component of vector *P*.





- Spatial Motion Description
  - $\checkmark$  Orientation of the whole rigid body
    - Consider pure rotation
    - Affix a moving frame {*B*} to the rigid body
    - Represent the rotation of two frames
  - ✓ Rotation Matrix

$${}^{A}R_{B} = \begin{bmatrix} {}^{A}\hat{x}_{B} \mid {}^{A}\hat{y}_{B} \mid {}^{A}\hat{z}_{B} \end{bmatrix} = \begin{bmatrix} {}^{r_{11}} & {}^{r_{12}} & {}^{r_{13}} \\ {}^{r_{21}} & {}^{r_{22}} & {}^{r_{23}} \\ {}^{r_{31}} & {}^{r_{32}} & {}^{r_{33}} \end{bmatrix}$$

Or  

$${}^{A}R_{B} = \begin{bmatrix} {}^{A}\hat{u} \mid {}^{A}\hat{v} \mid {}^{A}\hat{w} \end{bmatrix} = \begin{bmatrix} u_{x} & v_{x} & w_{x} \\ u_{y} & v_{y} & w_{y} \\ u_{z} & v_{z} & w_{z} \end{bmatrix}.$$



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- ✓ Rotation Matrix
  - Component wise:

$${}^{A}\hat{x}_{B} = {}^{A}\hat{u} = u_{x}\hat{i} + u_{y}\hat{j} + u_{z}\hat{k},$$
$${}^{A}\hat{y}_{B} = {}^{A}\hat{v} = v_{x}\hat{i} + v_{y}\hat{j} + v_{z}\hat{k},$$
$${}^{A}\hat{z}_{B} = {}^{A}\hat{w} = w_{x}\hat{i} + w_{y}\hat{j} + w_{z}\hat{k},$$

• Use dot product  $u_x = \hat{x}_B \cdot \hat{x}_A$  (in any frame)

$${}^{A}\mathbf{R}_{B} = \begin{bmatrix} {}^{A}\hat{\mathbf{x}}_{B} \mid {}^{A}\hat{\mathbf{y}}_{B} \mid {}^{A}\hat{\mathbf{z}}_{B} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{x}}_{B} \cdot \hat{\mathbf{x}}_{A} & \hat{\mathbf{y}}_{B} \cdot \hat{\mathbf{x}}_{A} & \hat{\mathbf{z}}_{B} \cdot \hat{\mathbf{x}}_{A} \\ \hat{\mathbf{x}}_{B} \cdot \hat{\mathbf{y}}_{A} & \hat{\mathbf{y}}_{B} \cdot \hat{\mathbf{y}}_{A} & \hat{\mathbf{z}}_{B} \cdot \hat{\mathbf{y}}_{A} \\ \hat{\mathbf{x}}_{B} \cdot \hat{\mathbf{z}}_{A} & \hat{\mathbf{y}}_{B} \cdot \hat{\mathbf{z}}_{A} & \hat{\mathbf{z}}_{B} \cdot \hat{\mathbf{z}}_{A} \end{bmatrix}.$$



- Direction Cosine representation
- The leading superscript is intentionally omitted, since the dot product is a scalar that can be evaluated in any frame {*A*} or {*B*}.

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- ✓ Rotation Matrix
  - Example 1:

a) Rotation about x axis with an angle of  $\alpha$ 

$${}^{A}R_{B} = R_{x}(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix},$$

b) Rotation about y axis with an angle of  $\beta$ 

$${}^{A}\!R_{B}=R_{y}(\beta)=\begin{bmatrix}\cos\beta & 0 & \sin\beta\\ 0 & 1 & 0\\ -\sin\beta & 0 & \cos\beta\end{bmatrix},$$



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- ✓ Rotation Matrix
  - Example 1: (Cont.)
    - c) Rotation about z axis with an angle of  $\gamma$





Property 1: Orthonormal

Rotation matrix is an orthonormal matrix, i.e.

 $u \cdot u = v \cdot v = w \cdot w = 1,$  $u \cdot v = v \cdot w = w \cdot u = 0.$ 

 $\{A\}$ 

{*B*}

z

OA

(c)

Rotation matrix: Nine Parameters with Six constraints.

Furthermore:  $u \times v = w$ ,  $v \times w = u$ ,  $w \times u = v$ .

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- ✓ Rotation Matrix Properties
  - Property 2: Transposition

By inspection: Rows of the rotation matrix  ${}^{A}\mathbf{R}_{B}$  are the unit vectors of frame {*A*} expressed in frame {*B*}:

$${}^{A}\mathbf{R}_{B} = \begin{bmatrix} A\hat{\mathbf{x}}_{B} \mid A\hat{\mathbf{y}}_{B} \mid A\hat{\mathbf{z}}_{B} \end{bmatrix} = \begin{bmatrix} {}^{B}\hat{\mathbf{x}}_{A}^{T} \\ {}^{B}\hat{\mathbf{y}}_{A}^{T} \\ {}^{B}\hat{\mathbf{z}}_{A}^{T} \end{bmatrix} \Rightarrow {}^{B}\mathbf{R}_{A} = {}^{A}\mathbf{R}_{B}^{T}.$$

Property 3: Inverse

The inverse of the rotation matrix is equal to its transpose

$${}^{B}\boldsymbol{R}_{A}={}^{A}\boldsymbol{R}_{B}^{-1}={}^{A}\boldsymbol{R}_{B}^{T}.$$

#### Note on the inverse of the rotation map.

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- ✓ Rotation Matrix Properties
  - Property 4: Pure rotation map For a pure rotation:



$$^{A}P = ^{A}R_{B}^{B}P$$

See the proof in reference book [4].

Example 2: Consider

$${}^{A}R_{B} = \begin{bmatrix} 0.933 & -0.067 & -0.354 \\ -0.067 & 0.933 & -0.354 \\ 0.354 & 0.354 & 0.866 \end{bmatrix} \text{ and } {}^{B}P = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}^{T}.$$
  
hen:  
$${}^{A}P = {}^{A}R_{B} {}^{B}P$$
$$= \begin{bmatrix} 0.933 & -0.067 & -0.354 \\ -0.067 & 0.933 & -0.354 \\ 0.354 & 0.354 & 0.866 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.93 \\ -1.07 \\ 0.35 \end{bmatrix}.$$

Norm of the vector in {*A*} and {*B*} frames are both  $\sqrt{5}$ , they are just rotated.

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 $\{A^{i}\}$ 

0

 $\{\boldsymbol{B}\}$ 



- ✓ Rotation Matrix Properties
  - Property 5: Determinant

$$\det({}^{A}R_{B}) = 1. \tag{2.19}$$

Proof:

$$det(^{A}R_{B}) = w_{x}(u_{y}v_{z} - v_{y}u_{z}) - w_{y}(u_{x}v_{z} - v_{x}u_{z}) + w_{x}(u_{x}v_{y} - v_{x}u_{y})$$
$$= w \cdot (u \times v)$$
$$= w \cdot w$$
$$= 1.$$

#### Norm of the vector under pure rotational map is preserved.

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- ✓ Rotation Matrix Properties
  - Property 6: Eigenvalues

Eigenvalues of a rotation matrix are: 1,  $e^{i\theta}$ ,  $e^{-i\theta}$ , where,

$$\theta = \cos^{-1} \frac{\operatorname{tr}({}^{A}R_{B}) - 1}{2}.$$
(2.20)

The angle of rotation in a pure rotational map is denoted by  $\theta$ .

What about the axis of Rotation?

It might be related to the real eigenvector of rotation matrix?!



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- Orientation Representation
  - ✓ Euler Angle-Axis of Rotation: Screw Axis
    - In rotation matrix representation some invariant parameters remain unchanged regardless of the frame of representation.
      - The pure rotation angle  $\theta$  found by (2.21)
      - The Euler axis of rotation is the real eigenvalue of the rotation matrix, represented by  $\hat{s}$ .
      - Euler or Screw Axis Representation  $s = \theta \hat{s}$

Four parameters:  $\{\hat{s}_x, \hat{s}_y, \hat{s}_z, \theta\}$ , while  $\|\hat{s}\| = 1$ .





- Orientation Representation
  - ✓ Screw Axis from Rotation Matrix
    - Given the rotation axis and angle:  $s = \theta \hat{s}$



Using Rodrigues's rotation formula

$$P_2 = P_1 \cos \theta + (\hat{s} \times P_1) \sin \theta + (P_1 \cdot \hat{s})(1 - \cos \theta)\hat{s}$$

Then the rotation matrix is found by:

$${}^{A}R_{B} = \begin{bmatrix} s_{x}^{2}v\theta + c\theta & s_{x}s_{y}v\theta - s_{z}s\theta & s_{x}s_{z}v\theta + s_{y}s\theta \\ s_{y}s_{x}v\theta + s_{z}s\theta & s_{y}^{2}v\theta + c\theta & s_{y}s_{z}v\theta - s_{x}s\theta \\ s_{z}s_{x}v\theta - s_{y}s\theta & s_{z}s_{y}v\theta + s_{x}s\theta & s_{z}^{2}v\theta + c\theta \end{bmatrix}$$

where,  $s\theta = \sin \theta$ ,  $c\theta = v\theta = 1 - \cos \theta$ .

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 $\{A\}$ 

0

 $\{\boldsymbol{B}\}$ 

Ζ



# Motion Description

- **Orientation Representation** 
  - ✓ Rotation Matrix from Screw Axis

Given  ${}^{A}\mathbf{R}_{B} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$ 

Euler or Screw Axis Representation  $s = \theta \hat{s}$ , where

$$s_x = \frac{r_{32} - r_{23}}{2s\theta},$$
  

$$\theta = \cos^{-1} \frac{\operatorname{tr}({}^A R_B) - 1}{2} \quad \text{and} \quad s_y = \frac{r_{13} - r_{31}}{2s\theta},$$
  

$$s_z = \frac{r_{21} - r_{12}}{2s\theta}.$$



where,  $\sin \theta = s \theta$ .

#### Or use the matrix eigenvector.

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- Orientation Representation
  - ✓ Euler Parameters or Unit Quaternion
    - Given the rotation axis and angle:  $s = \theta \hat{s}$
    - Four Euler parameters are defined as:

 $\epsilon_1 = s_x \sin(\theta/2); \quad \epsilon_2 = s_y \sin(\theta/2); \quad \epsilon_3 = s_z \sin(\theta/2); \quad \epsilon_4 = \cos(\theta/2).$  (2.77)

Unit Quaternion: 4 × 1 tuple

 $\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \end{bmatrix}^T$  in which  $\|\boldsymbol{\epsilon}\|^2 = \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = 1$ 

Very common in instrumentation systems.

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- Orientation Representation
  - ✓ Euler Parameters or Unit Quaternion
    - Given the quaternions:  $\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 & \epsilon_2 & \epsilon_3 & \epsilon_4 \end{bmatrix}^T$
    - The rotation matrix is found:

$${}^{A}R_{B} = \begin{bmatrix} 1 - 2\epsilon_{2}^{2} - 2\epsilon_{3}^{2} & 2(\epsilon_{1}\epsilon_{2} - \epsilon_{3}\epsilon_{4}) & 2(\epsilon_{1}\epsilon_{3} + \epsilon_{2}\epsilon_{4}) \\ 2(\epsilon_{1}\epsilon_{2} + \epsilon_{3}\epsilon_{4}) & 1 - 2\epsilon_{1}^{2} - 2\epsilon_{3}^{2} & 2(\epsilon_{2}\epsilon_{3} - \epsilon_{1}\epsilon_{4}) \\ 2(\epsilon_{1}\epsilon_{3} - \epsilon_{2}\epsilon_{4}) & 2(\epsilon_{2}\epsilon_{3} + \epsilon_{1}\epsilon_{4}) & 1 - 2\epsilon_{1}^{2} - 2\epsilon_{2}^{2} \end{bmatrix}.$$
 (2.78)

• Given The rotation matrix, the quaternions are found:





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# Scientist Bio



### **Leonhard Euler**

#### (15 April 1707 - 18 September 1783)

Was a Swiss mathematician, physicist, astronomer, geographer, logician and engineer who made important and influential discoveries in many branches of mathematics, such as infinitesimal calculus and graph theory, while also making pioneering contributions to several branches such as topology and analytic number theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function. He is also known for his work in mechanics, fluid dynamics, optics, astronomy and music theory rigid bodies.

Euler was one of the most eminent mathematicians of the 18th century and is held to be one of the greatest in history. He is also widely considered to be the most prolific, as his collected works fill 92 volumes, more than anyone else in the field.



A geometric interpretation of Euler's Formula

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- Orientation Representation
  - ✓ Euler Angles: Three parameter representation
    - Pitch-Roll-Yaw



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- ✓ Euler Angles:
  - Pitch-Roll-Yaw: rotation about fixed frame axes



Rotate moving frame about *x* with angle  $\alpha$  (pitch angle), then rotate about *y* with angle  $\beta$  (roll angle), and then rotate about *z* with angle  $\gamma$  (yaw angle)

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- ✓ Fixed Axes Euler Angles:
  - Pitch-Roll-Yaw: Pre-Multiplication

 $R_{PRY}(\alpha,\beta,\gamma)=R_z(\gamma)R_y(\beta)R_x(\alpha)$ 

$$= \begin{bmatrix} c\gamma & -s\gamma & 0\\ s\gamma & c\gamma & 0\\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta\\ 0 & 1 & 0\\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0\\ 0 & c\alpha & -s\alpha\\ 0 & s\alpha & c\alpha \end{bmatrix}, \quad (2.37)$$
$$= \begin{bmatrix} c\beta c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma\\ c\beta s\gamma & s\alpha s\beta s\gamma + c\alpha c\gamma & c\alpha s\beta s\gamma - s\alpha c\gamma\\ -s\beta & s\alpha c\beta & c\alpha c\beta \end{bmatrix}. \quad (2.38)$$
$$\text{Inverse Map:} \qquad \beta = \text{Atan2} (-r_{31}, \pm \sqrt{r_{11}^2 + r_{21}^2}), \qquad (x \neq 1) = \frac{1}{2} = \frac$$

#### where $atan(1.0/-1.0) = -\pi/4$ while $atan2(1.0, -1.0) = 3\pi/4$ .

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- ✓ Fixed Axes Euler Angles:
  - Pitch-Roll-Yaw:
  - Inverse Map: (if  $c\beta = 0$  or  $\beta = \pm \pi/2$ )

The rotation matrix is reduced to:

$$R_{PRY} = \begin{bmatrix} 0 & \pm s\alpha c\gamma - c\alpha s\gamma & \pm s\alpha c\gamma + c\alpha s\gamma \\ 0 & \pm s\alpha c\gamma + c\alpha s\gamma & \pm s\alpha c\gamma - c\alpha s\gamma \\ \mp 1 & 0 & 0 \end{bmatrix}$$

The inverse solution degenerates

Only the sum or difference of  $\alpha$  and  $\gamma$  may be computed:

$$\begin{cases} \beta = \pi/2 \\ \alpha - \gamma = \operatorname{Atan2} (r_{12}, r_{22}) \end{cases} \text{ or } \begin{cases} \beta = -\pi/2 \\ \alpha + \gamma = \operatorname{Atan2} (-r_{12}, r_{22}) \end{cases}$$

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- ✓ Euler Angles:
  - u v w: Rotation about moving frame axes



Rotate moving frame about u with angle  $\alpha$ , then rotate about v with angle  $\beta$ , and then rotate about w with angle  $\gamma$ .

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### ✓ Moving Axes Euler Angles:

• u - v - w: Post-Multiplication

 ${}^{A}R_{R}(\alpha,\beta,\gamma) = R_{u}(\alpha)R_{v}(\beta)R_{w}(\gamma).$ (2.42) $R_{uvw}(\alpha,\beta,\gamma) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{vmatrix} \begin{vmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{vmatrix} \begin{vmatrix} c\gamma & -s\gamma & 0 \\ s\gamma & c\gamma & 0 \\ 0 & 0 & 1 \end{vmatrix}$  $= \begin{bmatrix} c\beta c\gamma & -c\beta s\gamma & s\beta \\ s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta \\ -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta \end{bmatrix}.$ (2.43) Inverse Map: (if  $c\beta \neq 0$ )  $\beta = \text{Atan2}(r_{13}, \pm \sqrt{r_{11}^2 + r_{12}^2}),$  $\alpha = \operatorname{Atan2}\left(-r_{23}/c\beta, r_{33}/c\beta\right),$ 

 $\gamma = \operatorname{Atan2} \left( -r_{12}/c\beta, r_{11}/c\beta \right).$ 

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- ✓ Moving Axes Euler Angles:
  - u v w: Post-Multiplication
  - Inverse Map: (if  $c\beta = 0$ )

The inverse solution degenerates

Only the sum or difference of  $\alpha$  and  $\gamma$  may be computed:

$$\begin{cases} \beta = \pi/2 \\ \alpha + \gamma = \operatorname{Atan2} (r_{32}, r_{22}) \end{cases} \quad \text{or} \quad \begin{cases} \beta = -\pi/2 \\ \alpha - \gamma = \operatorname{Atan2} (r_{32}, r_{22}) \end{cases}$$

• Example:

For 
$$R = \begin{bmatrix} 0 & 0 & 1.0000 \\ 0.9658 & -0.2588 & 0 \\ 0.2588 & 0.9658 & 0 \end{bmatrix} \rightarrow c\beta = 0$$
, and  $\begin{cases} \beta = \pi/2 \\ \alpha + \gamma = 5\pi/12 \end{cases}$  or  $\begin{cases} \beta = -\pi/2 \\ \alpha - \gamma = 5\pi/12. \end{cases}$ 

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- ✓ Other Euler Angles: (24 Angle Set)
  - Moving frame: w v w

 $R_{wvw}(\alpha,\beta,\gamma) = R_w(\alpha)R_v(\beta)R_w(\gamma)$ 

	Γεαεβεγ — sαsγ	$-c\alpha c\beta s\gamma - s\alpha c\gamma$	cαsβ	
=	$s\alpha c\beta c\gamma + c\alpha s\gamma$	$-s\alpha c\beta s\gamma + c\alpha c\gamma$	sαsβ	
	$-s\beta c\gamma$	$s\beta s\gamma$	сβ	



Moving frame: w - u - w

$$\begin{split} R_{wuw}(\alpha,\beta,\gamma) &= R_w(\alpha) R_u(\beta) R_w(\gamma) \\ &= \begin{bmatrix} c\alpha c\gamma - s\alpha c\beta s\gamma & -c\alpha s\gamma - s\alpha c\beta c\gamma & s\alpha s\beta \\ s\alpha c\gamma + c\alpha c\beta s\gamma & -s\alpha s\gamma + c\alpha c\beta c\gamma & -c\alpha s\beta \\ s\beta s\gamma & s\beta c\gamma & c\beta \end{bmatrix}. \end{split}$$



All other set components in Appendix B.

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#### **Homogeneous Transformation**

Definition, consecutive transformation, inverse transformation, finite and infinitesimal angle rotation.

In this chapter we first briefly review the robot components, and then elaborate on spatial motion description. For this means, position and orientation representation by rotation matrix, screw axis, quaternions and Euler angles are introduces. General motion of a rigid body is then represented by Chasles's theorem, homogeneous transformation and screw axis representation.

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General Motion of a Rigid Body

Chasles' Theorem: The most general rigid body displacement can be produced by a translation along a line (called its screw axis) followed by a rotation about that same line.

- ✓ Simple Version:
  - General Motion = Rotation + Translation

 ${}^{A}P = {}^{A}R_{B} {}^{B}P + {}^{A}P_{O_{B}}.$ 

 $^{A}P = ^{A}T_{B} ^{B}P.$ 

Full matrix representation

Homogeneous Transformation Matrix  $(4 \times 4)$ :

$${}_{B}^{A}T = {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}P_{O_{B}} \\ \hline 0 & 0 & 0 & 1 \end{bmatrix}.$$



(2.53)

Then:

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- General Motion of a Rigid Body
  - ✓ Screw Displacement
    - General Motion =

Rotation about  $\hat{s}$  + Translation along  $\hat{s}$ 

 $\{\hat{\boldsymbol{s}}, \boldsymbol{\theta}\}$  +  $\{\boldsymbol{s_0}, d\}$ 

- The frame displacement <sup>A</sup>P<sub>OB</sub> is represented by s<sub>0</sub>, and The screw lead is denoted by d
- Eight parameter representation
   With two constraints:

$$\hat{\boldsymbol{s}}^T \hat{\boldsymbol{s}} = 1; \quad \boldsymbol{s}_o^T \hat{\boldsymbol{s}} = d.$$



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- General Motion of a Rigid Body
  - $\checkmark$  Given  ${}^{A}T_{B}$  Find Screw Parameters by:
    - Find  $\{\hat{s}, \theta\}$  from Slide 28 and find  $\{s_0, d\}$  by  $s_o = {}^{A}P_{O_B}$ ;  $d = s_o^T \hat{s}$ .
  - $\checkmark$  Given Screw Parameters Find  ${}^{A}T_{B}$  by:

$${}^{A}T_{B} = \begin{bmatrix} s_{x}^{2}v\theta + c\theta & s_{x}s_{y}v\theta - s_{z}s\theta & s_{x}s_{z}v\theta + s_{y}s\theta & p_{x} \\ s_{y}s_{x}v\theta + s_{z}s\theta & s_{y}^{2}v\theta + c\theta & s_{y}s_{z}v\theta - s_{x}s\theta & p_{y} \\ s_{z}s_{x}v\theta - s_{y}s\theta & s_{z}s_{y}v\theta + s_{x}s\theta & s_{z}^{2}v\theta + c\theta & p_{z} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
(2.64)

while,

$$p_{x} = d s_{x} - s_{o_{x}}(s_{x}^{2} - 1)\upsilon\theta - s_{o_{y}}(s_{x}s_{y}\upsilon\theta - s_{z}s\theta) - s_{o_{z}}(s_{x}s_{z}\upsilon\theta + s_{y}s\theta)$$

$$p_{y} = d s_{y} - s_{o_{x}}(s_{y}s_{x}\upsilon\theta + s_{z}s\theta) - s_{o_{y}}(s_{y}^{2} - 1)\upsilon\theta - s_{o_{z}}(s_{y}s_{z}\upsilon\theta - s_{x}s\theta)$$

$$p_{z} = d s_{z} - s_{o_{x}}(s_{z}s_{x}\upsilon\theta - s_{y}s\theta) - s_{o_{y}}(s_{z}s_{y}\upsilon\theta + s_{x}s\theta) - s_{o_{z}}(s_{z}^{2} - 1)\upsilon\theta$$

$$(2.65)$$

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- General Motion of a Rigid Body
  - ✓ Example: Given

$T_B =$	-0.2500	0.6124	+0.7500	-1.4142	•
4-	+0.7500	0.6124 0.5000	-0.2500 -0.6124	+1.4142 -1.5764	

The screw displacement is found as:

$$\left\{\theta = \frac{\pi}{3}; \quad \hat{s} = \frac{1}{\sqrt{2}} [1, 0, -1]^T\right\}$$

and

$$\left\{ d = 2; \quad s_0 = [1.4142, -1.5764, -1.4142]^T \right\}.$$

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# Scientist Bio



### **Michel Floréal Chasles**

#### (15 November 1793 - 18 December 1880)

Was a French mathematician. In 1837 he published the book *Aperçu historique sur l'origine et le développement des méthodes en géométrie* ("Historical view of the origin and development of methods in geometry"), a study of the method of reciprocal polars in projective geometry. The work gained him considerable fame and respect and he was appointed Professor at the École Polytechnique in 1841, then he was awarded a chair at the Sorbonne in 1846. In 1841, Charles published an English version as *Two Geometrical Memoirs on the General Properties of Cones of the Second Degree and on the Spherical Conics*, adding a significant amount of original material. In kinematics, Chasles's description of a Euclidean motion in space as screw displacement was seminal to the development of the theories of dynamics of rigid bodies.



Chasles' theorem

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### Contents

#### **Robot Components**

Links and joints, primary joints, compound joints, robot kinematic structures, Some serial robot structures.

#### **Spatial Motion Description**

Coordinate systems, position and orientation representation, rotation matrix, rotation matrix properties, screw axis, unit quaternion, Euler angles.

#### **General Motion of a Rigid Body**

Chasles' Theorem, rotation plus orientation, screw axis representation.

#### **Homogeneous Transformation**

Definition, consecutive transformation, inverse transformation, finite and infinitesimal angle rotation.

In this chapter we first briefly review the robot components, and then elaborate on spatial motion description. For this means, position and orientation representation by rotation matrix, screw axis, quaternions and Euler angles are introduces. General motion of a rigid body is then represented by Chasles's theorem, homogeneous transformation and screw axis representation.

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• General Motion of a Rigid Body



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- ✓ Transformation Arithmetic
  - Inverse Transformation

Direct inverse calculation of  ${}^{A}T_{B}$  might be computationally intensive.

$${}^{B}T_{A} = {}^{A}T_{B}^{-1} = \begin{bmatrix} {}^{A}R_{B}^{T} & {}^{-A}R_{B}^{T} {}^{A}P_{O_{B}} \\ \hline 0 & 0 & 0 & 1 \end{bmatrix}$$

Proof:

Recall  ${}^{B}R_{A} = {}^{A}R_{B}^{T}$ Furthermore,  ${}^{A}P_{O_{A}} = -{}^{A}P_{O_{B}}$   ${}^{B}P_{O_{A}} = {}^{B}R_{A} {}^{A}P_{O_{A}} = -{}^{B}R_{A} {}^{A}P_{O_{B}}$  $= -{}^{A}R_{B}^{T} {}^{A}P_{O_{B}}$ 

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- ✓ Transformation Arithmetic
  - Finite Angle Rotations

For finite angle of rotation the order of rotations are important.

 Example: Consider, first rotation about x̂ w Then, rotate about ẑ with θ<sub>2</sub>. The total rotation matrix is R<sub>1</sub>:

The order cannot be exchanged:  $R_1 \neq R_2$ 

$$R_{x}(\theta_{1}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{1} & -s_{1} \\ 0 & s_{1} & c_{1} \end{bmatrix},$$

$$R_{z}(\theta_{2}) = \begin{bmatrix} c_{2} & -s_{2} & 0 \\ s_{2} & c_{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_{1} = R_{x}(\theta_{1}) R_{z}(\theta_{2}) = \begin{bmatrix} c_{2} & -s_{2} & 0 \\ c_{1}s_{2} & c_{1}c_{2} & -s_{1} \\ s_{1}s_{2} & s_{1}c_{2} & c_{1} \end{bmatrix},$$

$$R_{2} = R_{z}(\theta_{2}) R_{x}(\theta_{1}) = \begin{bmatrix} c_{2} & -c_{1}s_{2} & s_{1}s_{2} \\ s_{2} & c_{1}c_{2} & -s_{1}c_{2} \\ 0 & s_{1} & c_{1} \end{bmatrix}.$$

Finite rotations obey matrix manipulation rules, and are not vector, and do not commute.

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- ✓ Transformation Arithmetic
  - Infinitesimal Angle Rotations

Infinitesimal angle of rotation act like vectors.

• Example:

Consider, first rotation about  $\hat{x}$  with  $\delta\theta_1$ , then, rotate about  $\hat{z}$  with  $\delta\theta_2$ . The total rotation matrix is  $R_1$ :  $R_1 = R_x(\theta_1) R_z(\theta_2) = \begin{bmatrix} 1 & -\delta\theta_2 & 0 \\ \delta\theta_2 & 1 & -\delta\theta_1 \\ \delta\theta_1\delta\theta_2 & \delta\theta_1 & 1 \end{bmatrix}$ , Exchange the order of rotation  $R_2$ :  $R_2 = R_z(\theta_2) R_x(\theta_1) = \begin{bmatrix} 1 & -\delta\theta_2 & \delta\theta_1\delta\theta_2 \\ \delta\theta_2 & 1 & -\delta\theta_1 \\ 0 & \delta\theta_1 & 1 \end{bmatrix}$ ,

For infinitesimal  $\delta \theta_i$ 's, the higher orders are close to zero, and the order can be exchanged:  $R_1 = R_2$ 

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- ✓ Example:
  - Consider the following rotation matrix is given

 ${}^{0}R_{1} = \begin{bmatrix} 0.933 & 0.167 & 0.354 \\ 0.067 & 0.933 & -0.354 \\ -0.354 & 0.354 & 0.866 \end{bmatrix}.$ 

While,  ${}^{0}P_{O_1} = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}^T$ ; Find  ${}^{0}T_1$ .

Solution: The given rotation matrix is not orthonormal, by inspection correct the mistyping error by:  $r_{12} = 0.067$ . Then

$${}^{1}T_{0} = {}^{0}T_{1}^{-1} = \begin{bmatrix} {}^{0}R_{1}^{T} & {}^{-0}R_{1}^{T} {}^{0}P_{O_{1}} \\ \hline 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0.933 & 0.067 & -0.354 & {}^{-1.80} \\ 0.067 & 0.933 & 0.354 & {}^{0.80} \\ 0.354 & -0.354 & 0.866 & {}^{-1.06} \\ \hline 0 & 0 & 0 & 1 \end{bmatrix}.$$

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#### **About Hamid D. Taghirad**

Hamid D. Taghirad has received his B.Sc. degree in mechanical engineering from <u>Sharif University of Technology</u>, Tehran, Iran, in 1989, his M.Sc. in mechanical engineering in 1993, and his Ph.D. in electrical engineering in 1997, both from <u>McGill University</u>, Montreal, Canada. He is currently the University Vice-Chancellor for <u>Global strategies and International Affairs</u>, Professor and the Director of the <u>Advanced Robotics and Automated System (ARAS)</u>, Department of Systems and Control, <u>Faculty of Electrical Engineering</u>, <u>K. N. Toosi University of Technology</u>, Tehran, Iran. He is a senior member of IEEE, and Editorial board of <u>International</u> <u>Journal of Robotics: Theory and Application</u>, and <u>International Journal of Advanced</u> <u>Robotic Systems</u>. His research interest is *robust* and *nonlinear control* applied to *robotic systems*. His publications include five books, and more than 250 papers in international Journals and conference proceedings.

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### **Robotics: Mechanics & Control**



### **Chapter 2: Motion Description**

To read more and see the course videos visit our course website: http://aras.kntu.ac.ir/arascourses/robotics/

# **Thank You**

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