Robotics: Mechanics & Control

Chapter 3: Kinematic Analysis

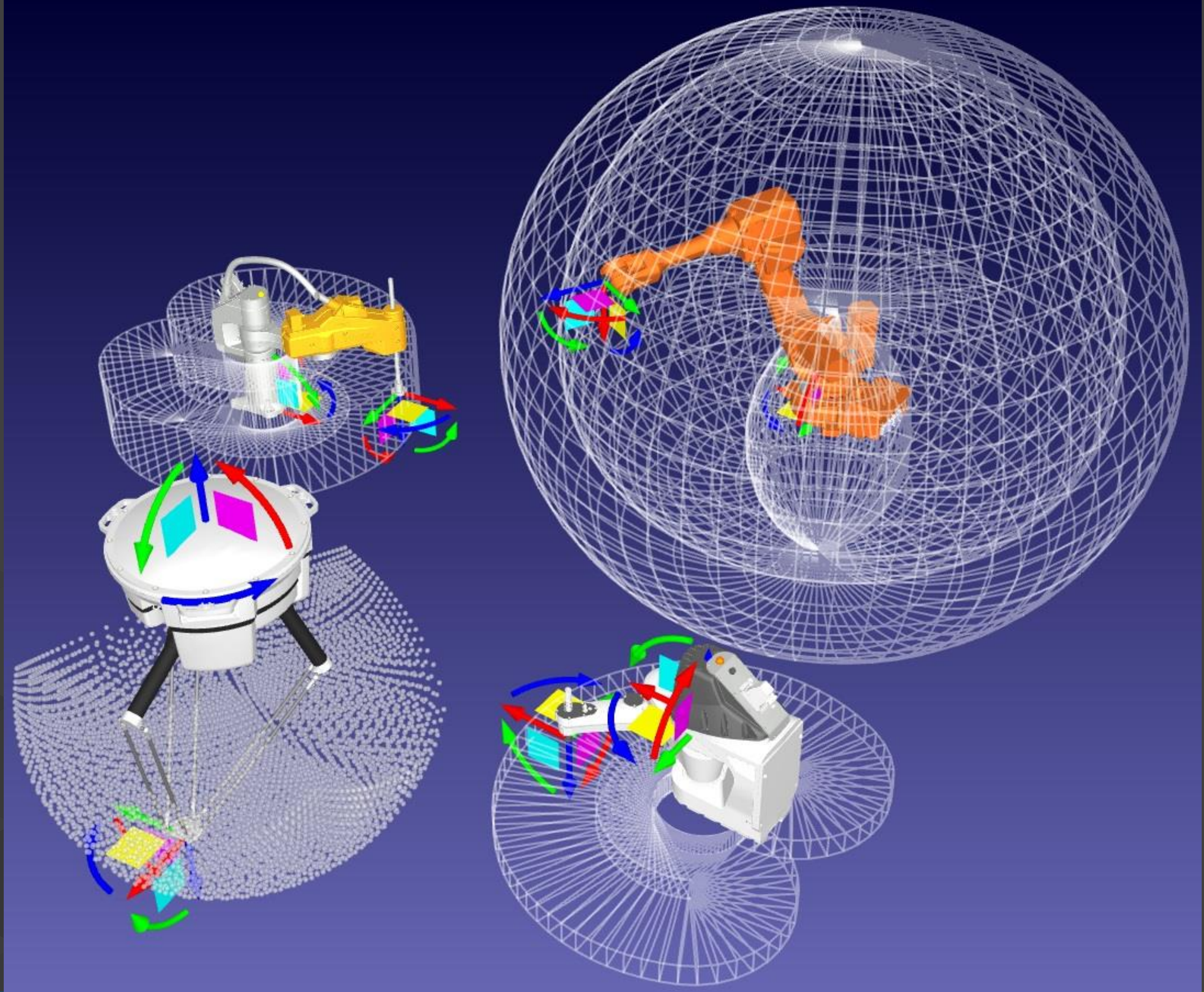
In this chapter we define the forward and inverse kinematic of serial manipulators. Different geometric, algorithmic, and screw-based solution methods will be examined, and Denavit-Hartenberg, and homogeneous transformation is introduced. The loop closure method in forward and inverse problem is solved for a number of case studies.

Welcome

To Your Prospect Skills

On Robotics :

Mechanics and Control



01 About ARAS

ARAS Research group originated in 1997 and is proud of its 22+ years of brilliant background, and its contributions to the advancement of academic education and research in the field of Dynamical System Analysis and Control in the robotics application. ***ARAS*** are well represented by the industrial engineers, researchers, and scientific figures graduated from this group, and numerous industrial and R&D projects being conducted in this group. The main asset of our research group is its human resources devoted all their time and effort to the advancement of science and technology. One of our main objectives is to use these potentials to extend our educational and industrial collaborations at both national and international levels. In order to accomplish that, our mission is to enhance the breadth and enrich the quality of our education and research in a dynamic environment.

Contents

In this chapter we define the forward and inverse kinematic of serial manipulators. Different geometric, algorithmic, and screw-based solution methods will be examined, and Denavit- Hartenberg, and homogeneous transformation is introduced. The loop closure method in forward and inverse problem is solved for a number of case studies.

Kinematic Analysis

- Definitions
  - ✓ The study of the geometry of motion in a robot, without considering the forces and torques that cause the motion.
  - ✓ A serial robot consist of
    - A single kinematic loop
    - A number of links and joints
    - The joints might be primary (P or R) or compound (U, C, S)
  - ✓ Kinematic loop closure
    - A loop consists of the consecutive links and joint to the end-effector
    - Rigid links with primary joints
    - Compound joints are reduced to a number of primary joints
    - The loop is written in a vector form
    - The joint motion variables form the Joint Space
    - The end-effector final motion DoF's form the Task Space

Kinematic Analysis

✓ Example: Elbow Manipulator

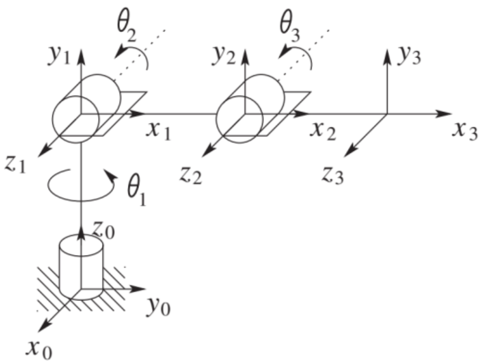
- 3DoF spatial manipulator (RRR)

Joint variables: $\boldsymbol{q} = [\theta_1 \quad \theta_2 \quad \theta_3]^T$

$$q_i = \begin{cases} \theta_i & \text{if joint } i \text{ is revolute} \\ d_i & \text{if joint } i \text{ is prismatic} \end{cases}$$

Task variables: $\boldsymbol{\chi} = [x_e \quad y_e \quad z_e]^T$

The position and orientation variables of end-effector

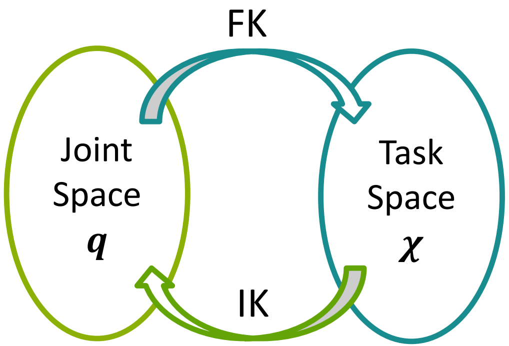


- Forward kinematics

Given $\boldsymbol{q}$ find $\boldsymbol{\chi}$

Inverse kinematics

Given $\boldsymbol{\chi}$ find $\boldsymbol{q}$



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In this chapter we define the forward and inverse kinematic of serial manipulators. Different geometric, algorithmic, and screw-based solution methods will be examined, and Denavit- Hartenberg, and homogeneous transformation is introduced. The loop closure method in forward and inverse problem is solved for a number of case studies.

Forward Kinematics

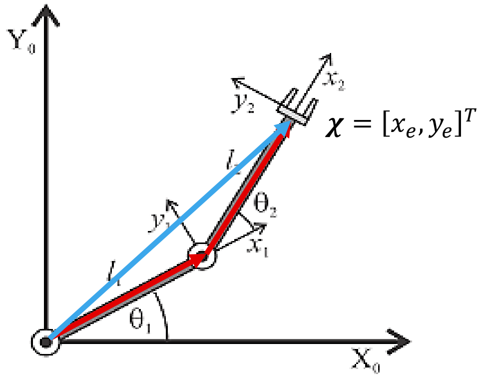
- Motivating Example
  - ✓ Kinematic loop closure
    - Assign base coordinate frame $\{0\}$

Denote $\boldsymbol{q} = [\theta_1, \theta_2]^T$ and $\boldsymbol{\chi} = [x_e, y_e]^T$

Denote the link vectors

and the end-effector vector

Write the loop closure vector equation:



$$\vec{l_1} + \vec{l_2} = \vec{\boldsymbol{\chi}}$$

$$l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) = x_e$$

$$l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) = y_e$$

Shorthand notation (FK)

$$l_1 c_1 + l_2 c_{12} = x_e$$

$$l_1 s_1 + l_2 s_{12} = y_e$$

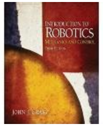
In which $c_1 = \cos\theta_1$, $s_1 = \sin\theta_1$, $c_{12} = \cos(\theta_1 + \theta_2)$, $s_{12} = \sin(\theta_1 + \theta_2)$.

Given joint variables $\boldsymbol{q} = [\theta_1, \theta_2]^T$, the task space variables $\boldsymbol{\chi} = [x_e, y_e]^T$ is found from FK formulation.

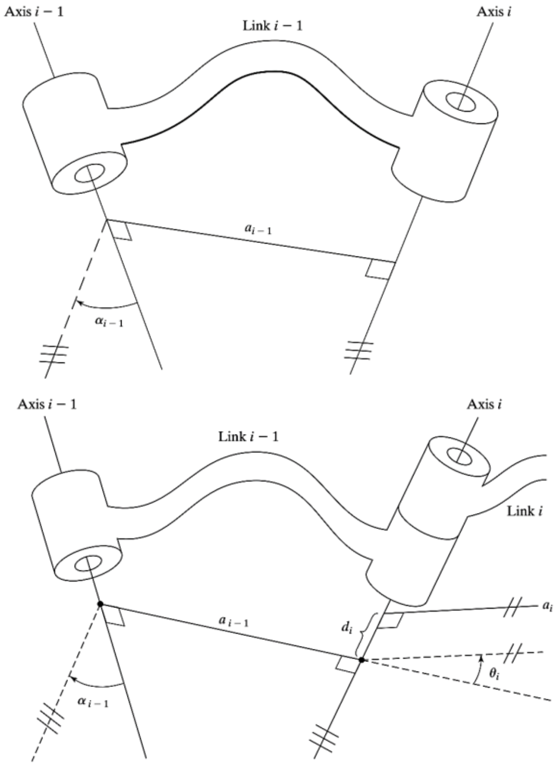
Inverse problem (IK) may be found by algebraic calculations.

Forward Kinematics

- Algorithmic Approach
  - ✓ General Link Parameters

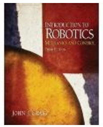


- Link length a_i and link twist α_i
  - Start from zero frame based on the joint axes
  - Link length a_{i-1} common normal line lengths
  - Link twist α_{i-1} relative angle of two joint axes
- Link offset d_i and joint angle θ_i
  - Neighboring link distance d_i and angle θ_i
  - For rotary joints $q_i = \theta_i$ is the joint variable
  - For prismatic joints $q_i = d_i$ is the joint variable
- First and Last link in the chain
  - Consider base of the robot the 0 link and frame
  - For the last frame on the end-effector coplanar to the previous frame

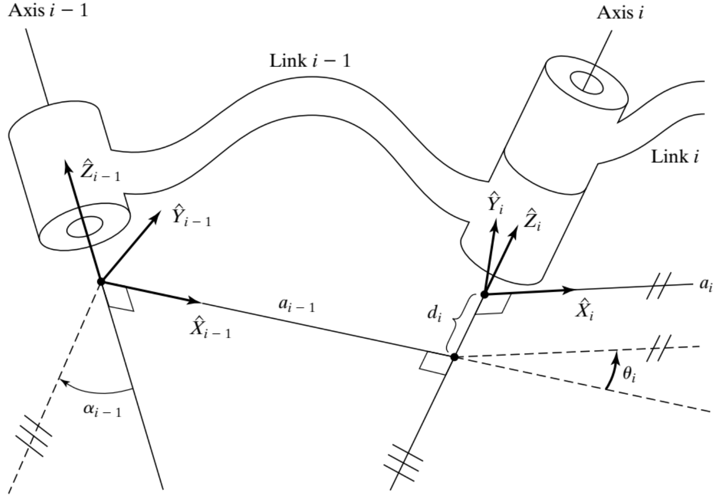


Forward Kinematics

- Algorithmic Approach
  - ✓ Frame Assignment (Craig's Convention)



    - The $\hat{Z}_i$ axis of rotation of $\{i\}$ joint (R)
    - OR the axis of translation of $\{i\}$ joint (P)
    - The origin of frame $\{i\}$ is at the intersection of perpendicular line to the axis i.
    - The $\hat{X}_i$ axis points along a_i along the common normal. In case a_i is zero $\hat{X}_i$ is normal to the plane of $\hat{Z}_i$ and $\hat{Z}_{i+1}$



  - ✓ Denavit-Hartenberg (DH) Parameters

a_i = The distance from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured along $\hat{X}_i$

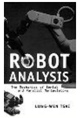
α_i = The angle from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured about $\hat{X}_i$

d_i = The distance from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured along $\hat{Z}_i$

θ_i = The angle from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured about $\hat{Z}_i$.

Forward Kinematics

- Algorithmic Approach



✓ Frame Assignment (Paul's Convention)

- The $\hat{Z}_{i-1}$ axis of rotation of $\{i\}$ joint (R)
- OR the axis of translation of $\{i\}$ joint (P)
- The origin of frame $\{i\}$ is at the intersection of perpendicular line to the axis i.
- The $\hat{X}_i$ axis points along a_i along the common normal. In case a_i is zero $\hat{X}_i$ is normal to the plane of $\hat{Z}_{i-1}$ and $\hat{Z}_i$

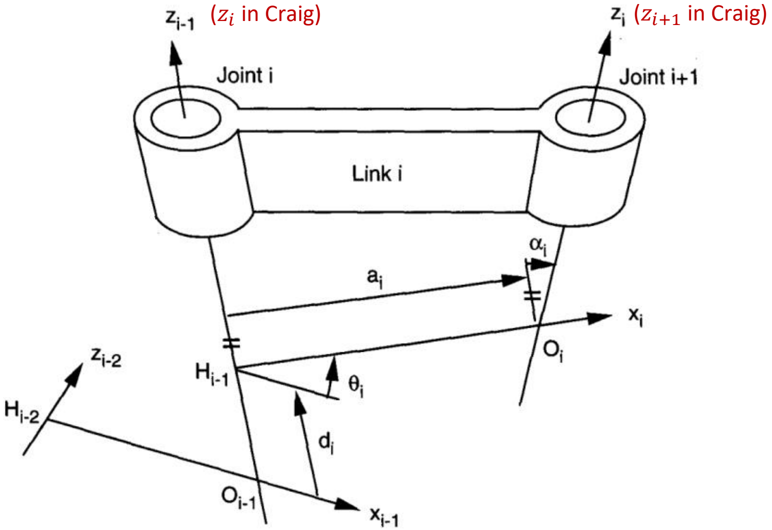
✓ Denavit-Hartenberg (DH) Parameters

$a_i =$ The distance from $\hat{Z}_{i-1}$ to $\hat{Z}_i$ measured along $\hat{X}_i$

$\alpha_i =$ The angle from $\hat{Z}_{i-1}$ to $\hat{Z}_i$ measured about $\hat{X}_i$

$d_i =$ The distance from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured along $\hat{Z}_{i-1}$

$\theta_i =$ The angle from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured about $\hat{Z}_{i-1}$.

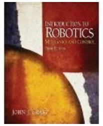


Forward Kinematics

- Algorithmic Approach
  - ✓ Denavit-Hartenberg (DH) Parameters

Forward Kinematics

- Algorithmic Approach
  - ✓ DH Homogeneous Transformations (Craig's Convention)



  - Consider three intermediate Frames

$$\{i-1\}, \qquad \{R\},\{Q\},\{P\}, \qquad \{i\}$$

The general transformation will be found by:
Considering four transformation about fixed Frames (post-multiplication)

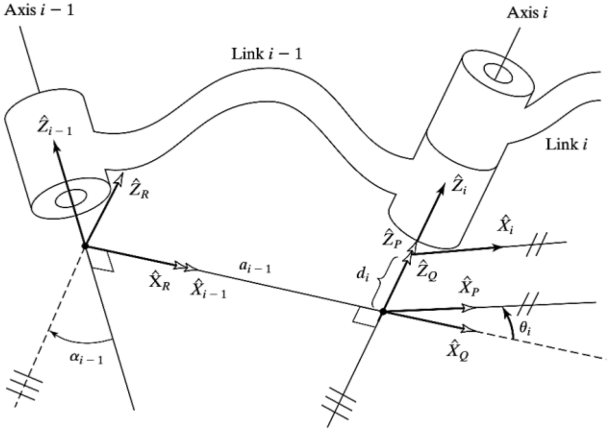
$$ {}^{i-1}_{i}T = {}^{i-1}_{R}T \; {}^{R}_{Q}T \; {}^{Q}_{P}T \; {}^{P}_{i}T. $$

In which,

$$ {}^{i-1}_{i}T = R_X(\alpha_{i-1}) D_X(a_{i-1}) R_Z(\theta_i) D_Z(d_i), $$

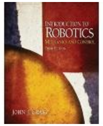
Or

$$ {}^{i-1}_{i}T = \mathrm{Screw}_X(a_{i-1}, \alpha_{i-1}) \; \mathrm{Screw}_Z(d_i, \theta_i), $$



Forward Kinematics

- Algorithmic Approach
  - ✓ DH Homogeneous Transformations (Craig's Convention)



$$
{}^{i-1}_{\ i}T = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & c\alpha_{i-1} & -s\alpha_{i-1} & 0 \\ 0 & s\alpha_{i-1} & c\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{i-1}_{\ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

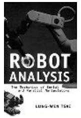
And the inverse is:

$$
{}^{\ \ i}_{i-1}T = \begin{bmatrix} c\theta_i & s\theta_i c\alpha_{i-1} & s\theta_i s\alpha_{i-1} & -a_{i-1} c\theta_i \\ -s\theta_i & c\theta_i c\alpha_{i-1} & c\theta_i s\alpha_{i-1} & -a_{i-1} s\theta_i \\ 0 & -s\alpha_{i-1} & c\alpha_{i-1} & -d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Forward Kinematics

- Algorithmic Approach

  ✓ DH Homogeneous Transformations (Paul’s Convention)



  - Consider three intermediate Frames

$$\{i-1\}, \qquad \{P\},\{Q\},\{R\}, \qquad \{i\}$$

The general transformation will be found by Considering four transformation about moving frames (pre-multiplication)

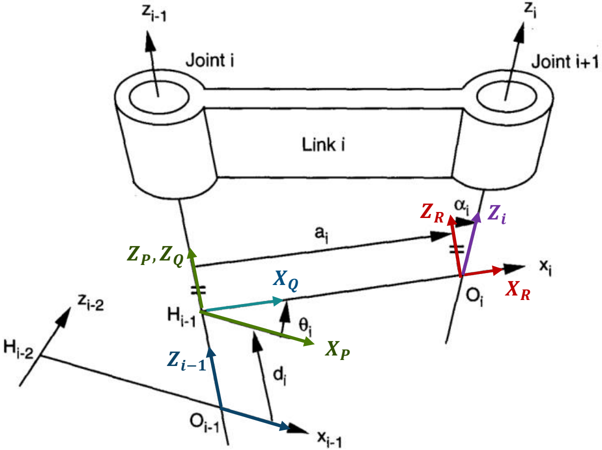
$$^{i-1}_{i}T = {}^{i-1}_{P}T \; {}^{P}_{Q}T \; {}^{Q}_{R}T \; {}^{R}_{i}T$$

In which,

$$^{i-1}_{i}T = D_z(d_i)R_z(\theta_i)D_x(a_i)R_x(\alpha_i)$$

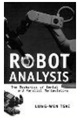
Or

$$^{i-1}_{i}T = \text{Screw}_z(d_i,\theta_i)\text{Screw}_x(a_i,\alpha_i)$$



Forward Kinematics

- Algorithmic Approach
  - ✓ DH Homogeneous Transformations (Paul's Convention)



$$ {}^{i-1}_{\ i}T = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{i-1}_{\ i}T = \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

And the inverse is

$$ {}^{\ \ \ i}_{i-1}T = \begin{bmatrix} c\theta_i & s\theta_i & 0 & -a_i \\ -s\theta_i c\alpha_i & c\theta_i c\alpha_i & s\alpha_i & -d_i s\alpha_i \\ s\theta_i s\alpha_i & -c\theta_i s\alpha_i & c\alpha_i & -d_i c\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Forward Kinematics

- Examples:
  - ✓ Example 1: Planar RRR Manipulator
    - Geometric Approach (3DoFs)

Denote $\boldsymbol{q} = [q_1, q_2, q_3]^T$ and $\boldsymbol{\chi} = [x_E, y_E, \Theta_E]^T$

Denote the link, and the end-effector vectors:

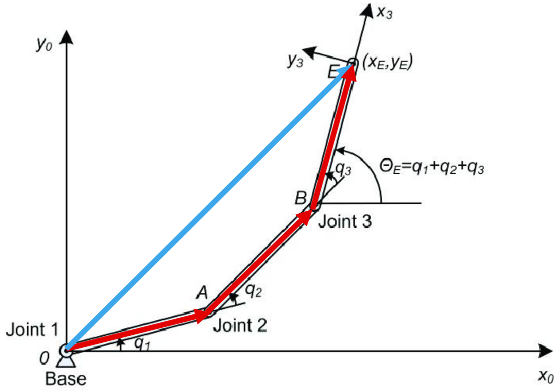
Write the loop closure vector equation:

$$\vec{l_1} + \vec{l_2} + \vec{l_3} = \vec{\chi}; \quad \Theta_E = q_1 + q_2 + q_3.$$

$$l_1 \cos q_1 + l_2 \cos(q_1 + q_2) + l_3 \cos(q_1 + q_2 + q_3) = x_E$$

$$l_1 \sin q_1 + l_2 \sin(q_1 + q_2) + l_2 \sin(q_1 + q_2 + q_3) = y_E$$

$$\Theta_E = q_1 + q_2 + q_3$$



Shorthand notation (FK)

$$l_1 c_1 + l_2 c_{12} + l_3 c_{123} = x_E$$

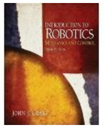
$$l_1 s_1 + l_2 s_{12} + l_3 s_{123} = y_E$$

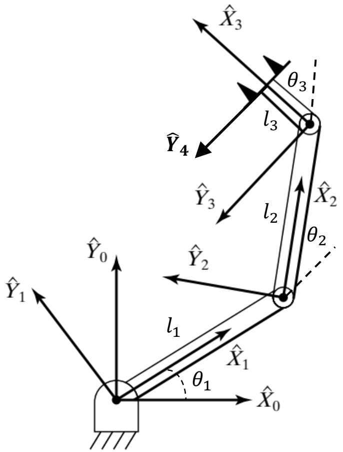
$$\Theta_E = q_1 + q_2 + q_3$$

In which $c_1 = \cos q_1, s_1 = \sin q_1, c_{12} = \cos(q_1 + q_2), s_{12} = \sin(q_1 + q_2), c_{123} = \cos(q_1 + q_2 + q_3), s_{123} = \sin(q_1 + q_2 + q_3)$.

Forward Kinematics

- Examples:
  - ✓ Example 1: Planar RRR Manipulator
    - Algorithmic Approach (Craig's Convention)





Denote $\boldsymbol{q} = [\theta_1, \theta_2, \theta_3]^T$ and $\boldsymbol{\chi} = [x_E, y_E, \Theta_E]^T$

Affix the frames and find DH-parameters.

Find the homogeneous transformations:

$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^2_3T = \begin{bmatrix} c_3 & -s_3 & 0 & l_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^3_4T = \begin{bmatrix} 1 & 0 & 0 & l_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

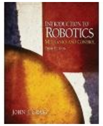
i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	0	l_1	0	θ_2
3	0	l_2	0	θ_3
4	0	l_3	0	0

Find the loop closure equation in matrix form:

$$ {}^0_ET = {}^0_4T = {}^0_1T \; {}^1_2T \; {}^2_3T \; {}^3_4T $$

Forward Kinematics

- Examples:
  - ✓ Example 1: (Cont.)
    - Algorithmic Approach (Craig's Convention)

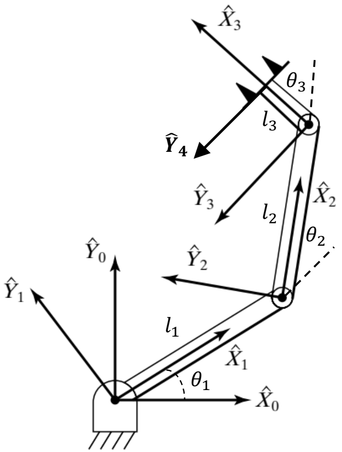


Calculate the loop closure matrix equation:

$$ {}^{0}_{E}T = {}^{0}_{4}T = {}^{0}_{1}T\ {}^{1}_{2}T\ {}^{2}_{3}T\ {}^{3}_{4}T $$

$$ {}^{0}_{E}T = \begin{bmatrix} c\Theta_E & -s\Theta_E & 0 & x_E \\ s\Theta_E & c\Theta_E & 0 & y_E \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

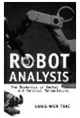
$$ {}^{0}_{4}T = {}^{0}_{1}T\ {}^{1}_{2}T\ {}^{2}_{3}T\ {}^{3}_{4}T = \begin{bmatrix} c_{123} & -s_{123} & 0 & l_1c_1 + l_2c_{12} + l_3c_{123} \\ s_{123} & c_{123} & 0 & l_1s_1 + l_2s_{12} + l_3s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$



Shorthand notation (FK)

$$ l_1c_1 + l_2c_{12} + l_3c_{123} = x_E $$
$$ l_1s_1 + l_2s_{12} + l_3s_{123} = y_E $$
$$ \Theta_E = \theta_1 + \theta_2 + \theta_3 $$

Forward Kinematics



- Examples:
  - ✓ Example 1: Planar RRR Manipulator
    - Algorithmic Approach (Paul's Convention)

Denote $\boldsymbol{q} = [\theta_1, \theta_2, \theta_3]^T$ and $\boldsymbol{\chi} = [x_E, y_E, \Theta_E]^T$

Affix the frames and find DH-parameters.

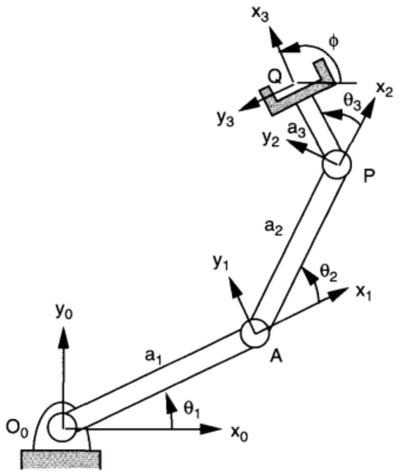
Find the homogeneous transformations:

$${}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^1_2T = \begin{bmatrix} c_2 & -s_2 & 0 & a_2 c_2 \\ s_2 & c_2 & 0 & a_2 s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$${}^2_3T = \begin{bmatrix} c_3 & -s_3 & 0 & a_3 c_3 \\ s_3 & c_3 & 0 & a_3 s_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find the loop closure equation in matrix form:

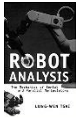
$${}^0_ET = {}^0_3T = {}^0_1T \; {}^1_2T \; {}^2_3T$$



i	α_i	a_i	d_i	θ_i
1	0	a_1	0	θ_1
2	0	a_2	0	θ_2
3	0	a_3	0	θ_3

Forward Kinematics

- Examples:
  - ✓ Example 1: (Cont.)
    - ▪ Algorithmic Approach (Paul's Convention)



Calculate the loop closure matrix equation:

$$ {}^{0}_{E}T = {}^{0}_{3}T = {}^{0}_{1}T \ {}^{1}_{2}T \ {}^{2}_{3}T $$

$$ {}^{0}_{E}T = \begin{bmatrix} c\Theta_E & -s\Theta_E & 0 & x_E \\ s\Theta_E & c\Theta_E & 0 & y_E \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

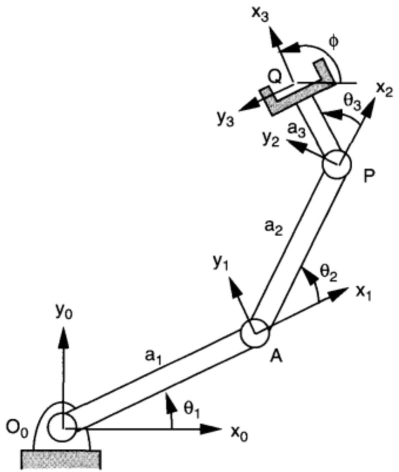
$$ {}^{0}_{3}T = {}^{0}_{1}T \ {}^{1}_{2}T \ {}^{2}_{3}T = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1c_1 + a_2c_{12} + a_3c_{123} \\ s_{123} & c_{123} & 0 & a_1s_1 + a_2s_{12} + a_3s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

Shorthand notation (FK)

$$ a_1c_1 + a_2c_{12} + a_3c_{123} = x_E $$

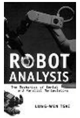
$$ a_1s_1 + a_2s_{12} + a_3s_{123} = y_E $$

$$ \Theta_E = \theta_1 + \theta_2 + \theta_3 $$



Forward Kinematics

- Examples:
  - ✓ Example 2: SCARA Manipulator



- Algorithmic Approach (Paul's Convention)

Denote $\boldsymbol{q} = [\theta_1, \theta_2, d_3, \theta_4]^T$ and $\boldsymbol{\chi} = [x_E, y_E, z_E, \Theta_E]^T$

Affix the frames and find DH-parameters.

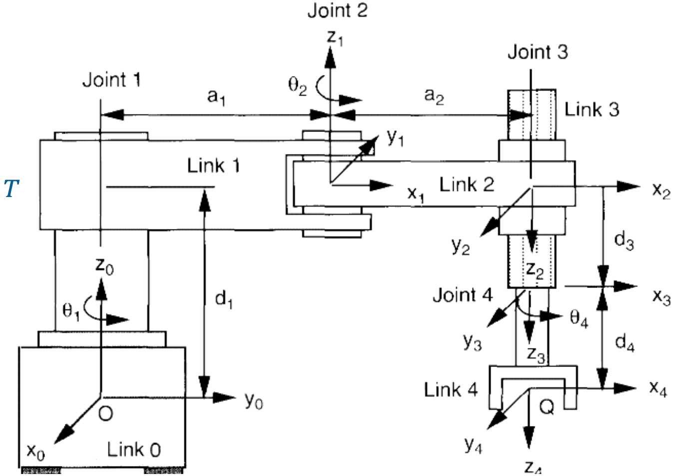
Find the homogeneous transformations:

$$
{}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^1_2T = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^2_3T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^3_4T = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Find the loop closure equation in matrix form:

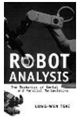
$$
{}^0_ET = {}^0_3T = {}^0_1T \; {}^1_2T \; {}^2_3T \; {}^3_4T
$$

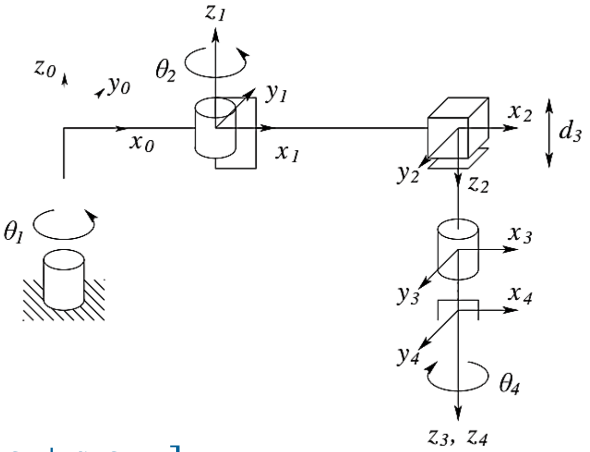


i	α_i	a_i	d_i	θ_i
1	0	a_1	d_1	θ_1
2	π	a_2	0	θ_2
3	0	0	d_3	0
4	0	0	d_4	θ_4

Forward Kinematics

- Examples:
  - ✓ Example 2: (Cont.)





▪ Algorithmic Approach (Paul's Convention)

Calculate the loop closure matrix equation:

$$ {}^{0}_{E}T = {}^{0}_{4}T = {}^{0}_{1}T\ {}^{1}_{2}T\ {}^{2}_{3}T\ {}^{3}_{4}T $$

$$ {}^{0}_{E}T = \begin{bmatrix} c\Theta_E & -s\Theta_E & 0 & x_E \\ s\Theta_E & c\Theta_E & 0 & y_E \\ 0 & 0 & 1 & z_E \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{4}T = {}^{0}_{1}T\ {}^{1}_{2}T\ {}^{2}_{3}T\ {}^{3}_{4}T = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & d_1 - d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

Shorthand notation (FK)

$$ a_1c_1 + a_2c_{12} = x_E, \qquad z_E = d_1 - d_3 - d_4, $$
$$ a_1s_1 + a_2s_{12} = y_E, \qquad -\Theta_E = -\theta_1 - \theta_2 + \theta_4. $$

Forward Kinematics

- Examples:
  - ✓ Example 3: Cylindrical Robot (RPP)

  - Algorithmic Approach (Paul's Convention)

Denote $\boldsymbol{q} = [\theta_1, d_2, d_3\,]^T$ and $\boldsymbol{\chi} = [x_E, z_E, \Theta_E]^T$

Affix the frames and find DH-parameters.

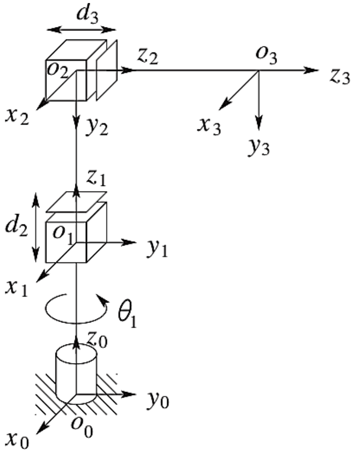
Find the homogeneous transformations:

$$ {}^0_1T = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^1_2T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^2_3T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

Find the loop closure equation in matrix form:

$$ {}^0_ET = {}^0_3T = {}^0_1T\ {}^1_2T\ {}^2_3T $$



i	α_i	a_i	d_i	θ_i
1	0	0	d_1	θ_1
2	$-\pi/2$	0	d_2	0
3	0	0	d_3	0

Forward Kinematics

- Examples:
  - ✓ Example 3: (Cont.)

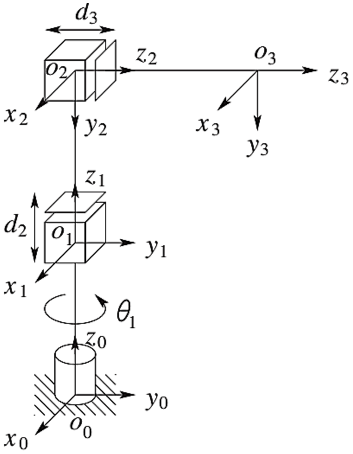
  - ▪ Algorithmic Approach (Paul's Convention)

Calculate the loop closure matrix equation:

$$ {}^{0}_{E}T = {}^{0}_{3}T = {}^{0}_{1}T \ {}^{1}_{2}T \ {}^{2}_{3}T $$

$$ {}^{0}_{E}T = \begin{bmatrix} c\Theta_E & -s\Theta_E & 0 & x_E \\ s\Theta_E & c\Theta_E & 0 & y_E \\ 0 & 0 & 1 & z_E \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{4}T = {}^{0}_{1}T \ {}^{1}_{2}T \ {}^{2}_{3}T = \begin{bmatrix} c_1 & 0 & -s_1 & -s_1 d_3 \\ s_1 & 0 & c_1 & c_1 d_3 \\ 0 & -1 & 0 & d_1 + d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$



Shorthand notation (FK)

$$ -s_1 d_3 = x_E, \qquad z_E = d_1 + d_2, $$
$$ c_1 d_3 = y_E, \qquad \Theta_E = f(\theta_1). $$

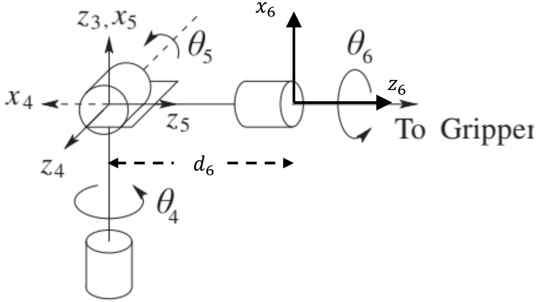
Forward Kinematics

- Examples:
  - ✓ Example 4: Spherical Wrist (RRR)
    - Algorithmic Approach (Paul’s Convention)

Denote $\boldsymbol{q} = [\theta_4, \theta_5, \theta_5]^T$ and $\boldsymbol{\chi} = [\phi, \theta, \psi]^T$

Affix the frames and find DH-parameters.

Find the homogeneous transformations:



i	α_i	a_i	d_i	θ_i
4	$-\pi/2$	0	0	θ_4
5	$\pi/2$	0	0	θ_5
6	0	0	d_6	θ_6

$$ {}^3_4T = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^4_5T = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

$$ {}^5_6T = \begin{bmatrix} c_6 & 0 & -s_6 & 0 \\ s_6 & 0 & c_6 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

Find the loop closure equation in matrix form:

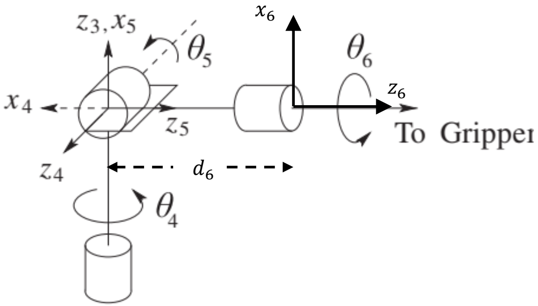
$$ {}^3_ET = {}^3_6T = {}^3_4T \; {}^4_5T \; {}^5_6T $$

Forward Kinematics

- Examples:

  ✓ Example 4: (Cont.)

  - Algorithmic Approach (Paul's Convention)



Calculate the loop closure matrix equation:

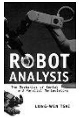
$$ {}^3_E T = {}^3_6 T = {}^3_4 T \; {}^4_5 T \; {}^5_6 T $$

$$ {}^3_E T = \begin{bmatrix} R(\phi, \theta, \psi) & d^3_E \\ 0 & 1 \end{bmatrix} $$

$$ {}^3_6 T = {}^3_4 T \; {}^4_5 T \; {}^5_6 T = \begin{bmatrix} c_4 c_5 c_6 - s_4 s_6 & -c_4 c_5 s_6 - s_4 c_6 & c_4 s_5 & c_4 s_5 d_6 \\ s_4 c_5 c_6 + c_4 s_6 & -s_4 c_5 s_6 + c_4 c_6 & s_4 s_5 & s_4 s_5 d_6 \\ -s_5 c_6 & s_5 s_6 & c_5 & c_5 d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Forward Kinematics

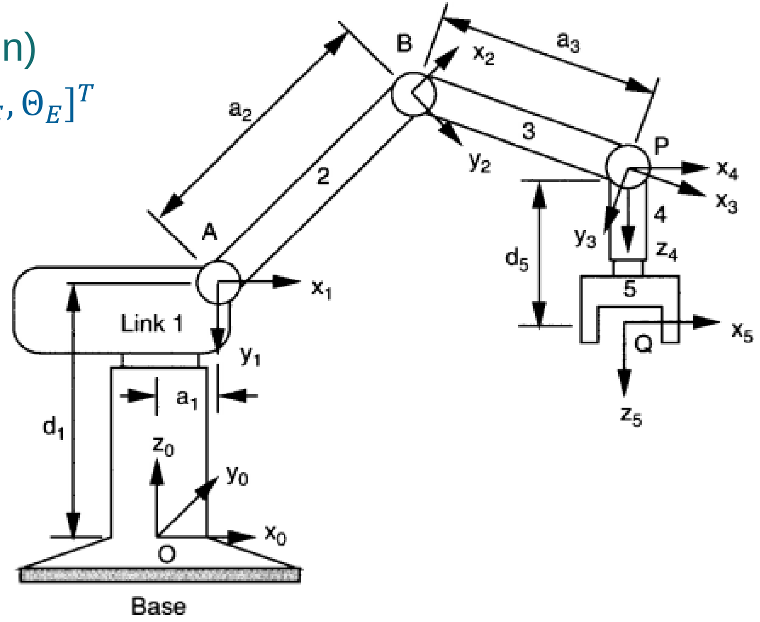
- Examples:
  - ✓ Example 5: Scorbot 5R Manipulator
    - ▪ Algorithmic Approach (Paul's Convention)



Denote $\boldsymbol{q} = [\theta_1, \theta_2, \theta_3, \theta_4, \theta_5]^T$ and $\boldsymbol{\chi} = [\boldsymbol{x}_E, \Theta_E]^T$

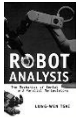
Affix the frames and find DH-parameters.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	a_1	d_1	θ_1
2	0	a_2	0	θ_2
3	0	a_3	0	θ_3
4	$-\pi/2$	0	0	θ_4
5	0	0	d_5	θ_5



Forward Kinematics

- Examples:
  - ✓ Example 5: (Cont.)

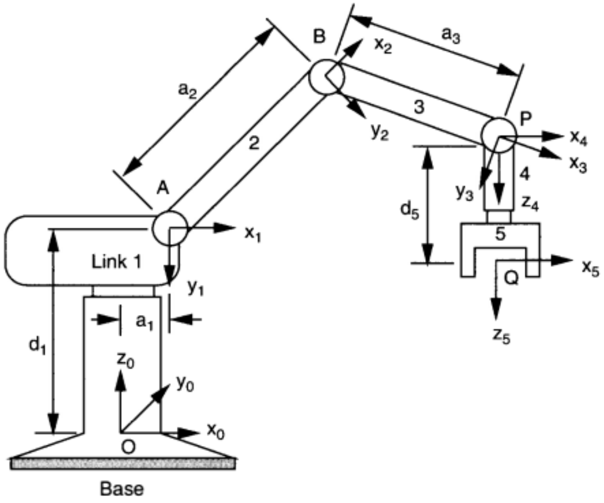


- Algorithmic Approach (Paul's Convention)

Calculate the loop closure matrix equation:

$$ {}^{0}_{E}T = {}^{0}_{5}T = A_1\ A_2\ A_3\ A_4\ A_5 $$

$$ {}^{0}_{E}T = \begin{bmatrix} R(\boldsymbol{\Theta}_E) & \boldsymbol{x}_E^0 \\ 0 & 1 \end{bmatrix} $$

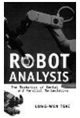


$$ {}^{0}A_1 = \begin{bmatrix} c\theta_1 & 0 & -s\theta_1 & a_1 c\theta_1 \\ s\theta_1 & 0 & c\theta_1 & a_1 s\theta_1 \\ 0 & -1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^{1}A_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

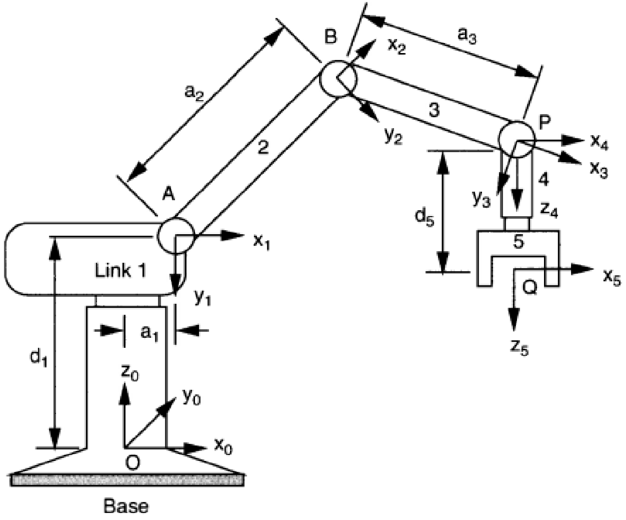
$$ {}^{2}A_3 = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & a_3 c\theta_3 \\ s\theta_3 & c\theta_3 & 0 & a_3 s\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^{3}A_4 = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & 0 \\ s\theta_4 & 0 & c\theta_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^{4}A_5 = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & 0 \\ s\theta_5 & c\theta_5 & 0 & 0 \\ 0 & 0 & 1 & d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

Forward Kinematics

✓ Example 5: (Cont.)



$$^{0}A_n = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{q} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$u_x = c\theta_1 c\theta_{234} c\theta_5 + s\theta_1 s\theta_5,$$

$$u_y = s\theta_1 c\theta_{234} c\theta_5 - c\theta_1 s\theta_5,$$

$$u_z = -s\theta_{234} c\theta_5,$$

$$v_x = -c\theta_1 c\theta_{234} s\theta_5 + s\theta_1 c\theta_5,$$

$$v_y = -s\theta_1 c\theta_{234} s\theta_5 - c\theta_1 c\theta_5,$$

$$v_z = s\theta_{234} s\theta_5,$$

$$w_x = -c\theta_1 s\theta_{234},$$

$$w_y = -s\theta_1 s\theta_{234},$$

$$w_z = -c\theta_{234},$$

$$q_x = c\theta_1 (a_1 + a_2 c\theta_2 + a_3 c\theta_{23} - d_5 s\theta_{234}),$$

$$q_y = s\theta_1 (a_1 + a_2 c\theta_2 + a_3 c\theta_{23} - d_5 s\theta_{234}),$$

$$q_z = d_1 - a_2 s\theta_2 - a_3 s\theta_{23} - d_5 c\theta_{234}.$$

Forward Kinematics

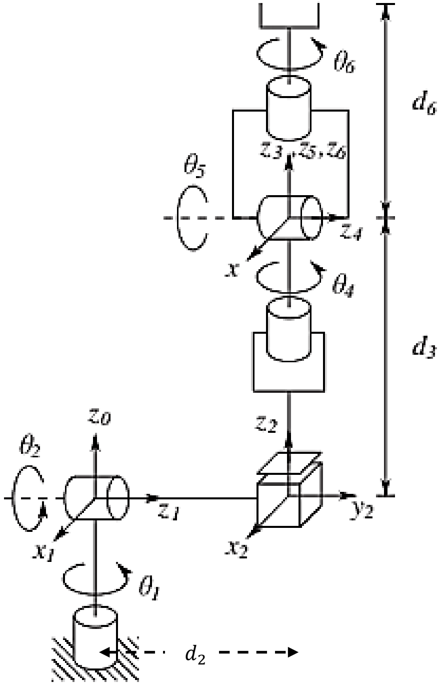
- Examples:
  - ✓ Example 6: Stanford Manipulator (2RP3R)

- Algorithmic Approach (Paul's Convention)

Denote $\boldsymbol{q} = [\theta_1, \theta_2, d_3, \theta_4, \theta_5, \theta_6]^T$ and $\boldsymbol{\chi} = [\boldsymbol{x_E}, \boldsymbol{\Theta_E}]^T$

Affix the frames and find DH-parameters.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	0	θ_1
2	$\pi/2$	0	d_2	θ_2
3	0	0	d_3	0
4	$-\pi/2$	0	0	θ_4
5	$\pi/2$	0	0	θ_5
6	0	0	d_6	θ_6



Forward Kinematics

- Examples:
  - ✓ Example 6: (Cont.)
    - Algorithmic Approach (Paul's Convention)

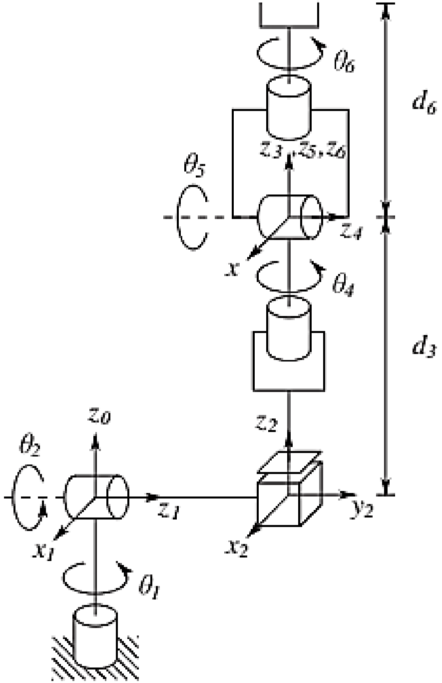
Calculate the loop closure matrix equation:

$${}^0_ET = {}^0_6T = A_1\,A_2\,A_3\,A_4\,A_5\,A_6$$

$${}^0_ET = \begin{bmatrix} R(\boldsymbol{\Theta}_E) & \boldsymbol{x}_E^0 \\ 0 & 1 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



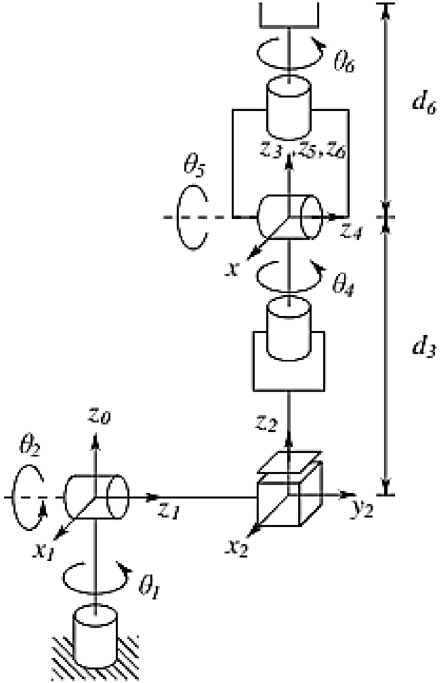
Forward Kinematics

✓ Example 6: (Cont.)

$$T_6^0 = A_1 \cdots A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

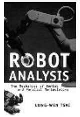
in which

$$\begin{aligned}
r_{11} &= c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) \\
r_{21} &= s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) \\
r_{31} &= -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 \\
r_{12} &= c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) \\
r_{22} &= -s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) \\
r_{32} &= s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 \\
r_{13} &= c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 \\
r_{23} &= s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 \\
r_{33} &= -s_2c_4s_5 + c_2c_5 \\
d_x &= c_1s_2d_3 - s_1d_2 + +d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) \\
d_y &= s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) \\
d_z &= c_2d_3 + d_6(c_2c_5 - c_4s_2s_5)
\end{aligned}$$



Forward Kinematics

- Successive Screw Method



  ✓ Use Screw Displacement Representation

    - Screw axis $\hat{\boldsymbol{s}}$ along

      the rotation axis in R joints

      the translation axis in P joints

    - The screw displacement $\boldsymbol{s_0}$ In base frame.

      Consider the screw representation $\hat{\$}_{\boldsymbol{i}}$ in home configuration

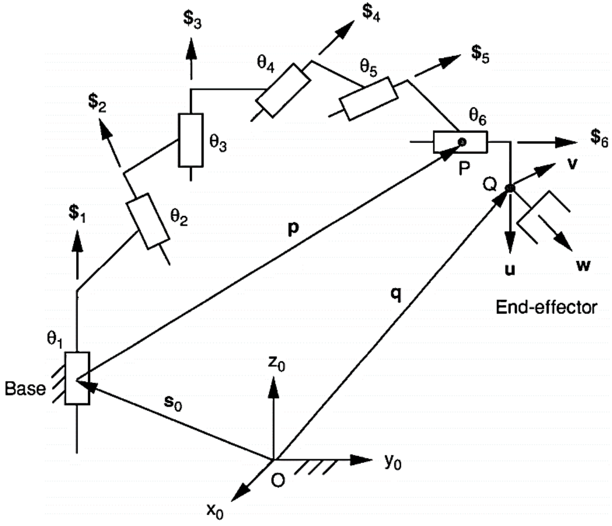
      Find the target screw axis by consecutive screw displacement

    - Find the Homogeneous transformations $A_i$from screw representation $\hat{\$}_{\boldsymbol{i}}$

      The joint variables θ_i and d_i are used in the homogeneous transformations

    - Apply the loop closure equation by $A_E = A_1\, A_2 \cdots A_{n-1}\, A_n$

      No Frame assignment; just the base and the end effector frames are needed



Forward Kinematics

- Successive Screw Method
  - ✓ Review Screw Displacement Formulation
    - General Motion =

      Rotation about $\hat{\boldsymbol{s}}$ + Translation along $\hat{\boldsymbol{s}}$

      $$\{\hat{\boldsymbol{s}}, \theta\} \qquad + \qquad \{\boldsymbol{s_0}, d\}$$

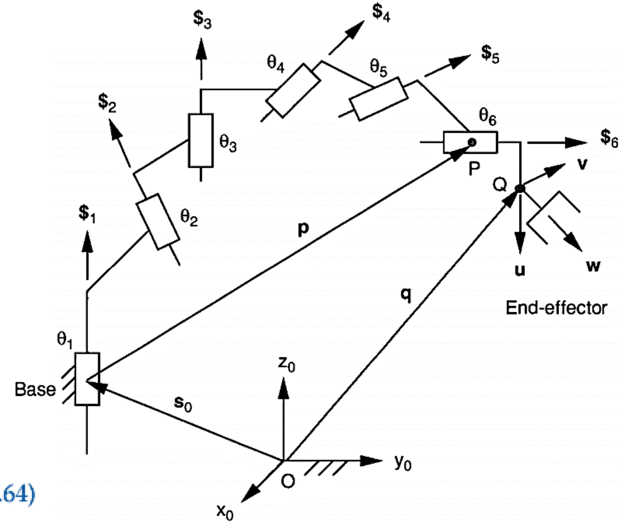
      where $\quad \hat{\mathbf{s}}^T\hat{\mathbf{s}} = 1; \quad \mathbf{s}_o^T\hat{\mathbf{s}} = d.$

    - Given Screw Parameters Find A_i by:

$$A_i = \left[\begin{array}{ccc|c} s_x^2 v\theta + c\theta & s_x s_y v\theta - s_z s\theta & s_x s_z v\theta + s_y s\theta & p_x \\ s_y s_x v\theta + s_z s\theta & s_y^2 v\theta + c\theta & s_y s_z v\theta - s_x s\theta & p_y \\ s_z s_x v\theta - s_y s\theta & s_z s_y v\theta + s_x s\theta & s_z^2 v\theta + c\theta & p_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \qquad (2.64)$$

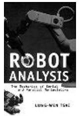
while,

$$\begin{aligned} p_x &= d\, s_x - s_{o_x}(s_x^2 - 1)v\theta - s_{o_y}(s_x s_y v\theta - s_z s\theta) - s_{o_z}(s_x s_z v\theta + s_y s\theta) \\ p_y &= d\, s_y - s_{o_x}(s_y s_x v\theta + s_z s\theta) - s_{o_y}(s_y^2 - 1)v\theta - s_{o_z}(s_y s_z v\theta - s_x s\theta) \\ p_z &= d\, s_z - s_{o_x}(s_z s_x v\theta - s_y s\theta) - s_{o_y}(s_z s_y v\theta + s_x s\theta) - s_{o_z}(s_z^2 - 1)v\theta \end{aligned} \qquad (2.65)$$



Forward Kinematics

- Successive Screw Method



✓ Recipe

- Consider the manipulator in Reference Position

  Where the joint variables are all zero $\theta_i = d_i = 0$

  Determine the end effector position $\boldsymbol{x}_E^{\boldsymbol{0}}$

  and orientation $\boldsymbol{R}_E^0 = [\hat{\boldsymbol{u}}_0 \quad \hat{\boldsymbol{v}}_0 \quad \hat{\boldsymbol{w}}_0]$

  And the Screw axis representations for n joints

  $$\hat{\$}_i = \{\hat{\boldsymbol{s}}_i, \boldsymbol{s}_{o_i}\} \text{ for } \quad i = 1,2,\dots,n.$$

- Consider the manipulator in target Position

  Determine the end effector position $\boldsymbol{x}_E$ and orientation $\boldsymbol{R}_E = [\hat{\boldsymbol{u}} \quad \hat{\boldsymbol{v}} \quad \hat{\boldsymbol{w}}]$

  This is found based on task space variables $\boldsymbol{\chi}$.

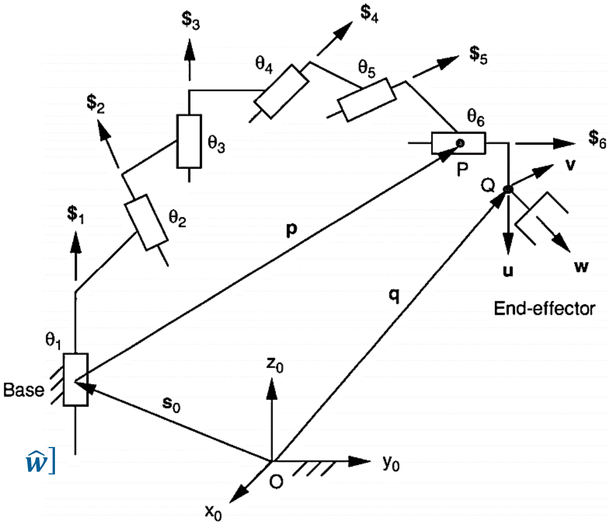
- Apply the loop closure equation

  Calculate the homogeneous transformation of $\hat{\$}_i$

  Determine the target screw axis by consecutive screw displacement

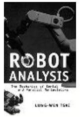
  $$A_E = A_1\, A_2 \cdots A_{n-1}\, A_n \rightarrow \boldsymbol{x}_E = A_1\, A_2 \cdots A_{n-1}\, A_n\, \boldsymbol{x}_E^{\boldsymbol{0}}$$

- In forward kinematics given $\boldsymbol{q}$ the task space variable $\boldsymbol{\chi}$ is found.
- In inverse kinematics given $\boldsymbol{\chi}$ the joint space variable $\boldsymbol{q}$ is found.



Forward Kinematics

- Successive Screw Method



✓ Example 1

- Consider Planar RRR Manipulator
- Reference Pose:

Consider all the joint variables $\theta_i = 0$.

Consider the base frame $x - y - z$ and

The end effector frame $u - v - w$

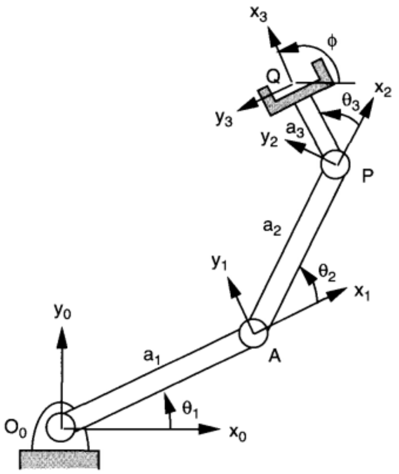
Derive the reference pose by:

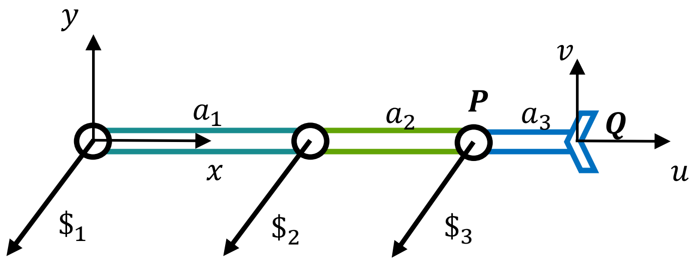
$$\boldsymbol{u}_0 = [1, 0, 0]^T, \boldsymbol{v}_0 = [0, 1, 0]^T, \boldsymbol{w}_0 = [0, 0, 1]^T$$

$$\boldsymbol{x}_E^0 = \boldsymbol{q_0} = [a_1 + a_2 + a_3, 0, 0]^T$$

And for the wrist point $\boldsymbol{P}$

$$\boldsymbol{p_0} = [a_1 + a_2, 0, 0]^T.$$





Forward Kinematics

- Successive Screw Method

✓ Example 1: (Cont.)

- Reference Pose:

Denote the screws $\hat{\$}_i$ as on the diagram

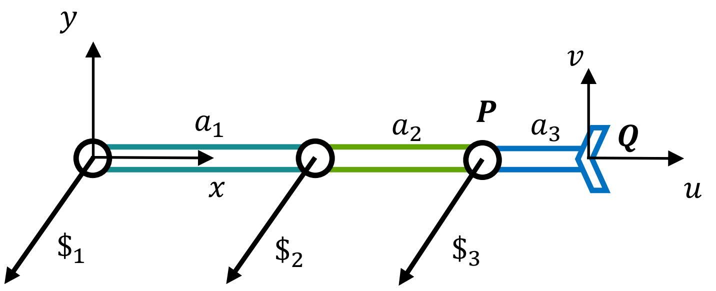
Find the screw parameters $\hat{\boldsymbol{s}}_{\boldsymbol{i}}$ and $\boldsymbol{s_{o}}_{\boldsymbol{i}}$ as given in the table

- Target Pose:

Let the target position of the wrist be:

$$\mathbf{u} = [u_x, u_y, u_z]^{\mathrm{T}}, \quad \mathbf{v} = [v_x, v_y, v_z]^{\mathrm{T}}, \quad \mathbf{w} = [w_x, w_y, w_z]^{\mathrm{T}},$$
$$\mathbf{p} = [p_x, p_y, p_z]^{\mathrm{T}}.$$

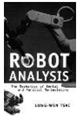
And for the end effector $\boldsymbol{q} = \left[q_x, q_y, q_z\right]^T$.



Join i	$\boldsymbol{s_i}$	$\boldsymbol{s_{0}}_{\boldsymbol{i}}$
1	$(0,0,1)$	$(0,0,0)$
2	$(0,0,1)$	$(a_1,0,0)$
3	$(0,0,1)$	$(a_1+a_2,0,0)$

Forward Kinematics

- Successive Screw Method



✓ Example 1: (Cont.)

▪ Screw Transformation Matrices:

$$A_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_1 v\theta_2 \\ s\theta_2 & c\theta_2 & 0 & -a_1 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad A_2 = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & (a_1+a_2)v\theta_3 \\ s\theta_3 & c\theta_3 & 0 & -(a_1+a_2)s\theta_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$A_1A_2A_3 = \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & a_1c\theta_1 + a_2c\theta_{12} - (a_1+a_2)c\theta_{123} \\ s\theta_{123} & c\theta_{123} & 0 & a_1s\theta_1 + a_2s\theta_{12} - (a_1+a_2)s\theta_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

Loop closure equation: $\boldsymbol{q} = A_1A_2A_3\boldsymbol{q_0}$;

$$q_x = a_1c\theta_1 + a_2c\theta_{12} + a_3c\theta_{123}$$

$$q_y = a_1s\theta_1 + a_2s\theta_{12} + a_3s\theta_{123}$$

$$q_z = 0$$

The results is the same as before.

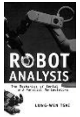
Orientation Equivalence:

$${}^0_ER = \begin{bmatrix} c\Theta_E & -s\Theta_E & 0 \\ s\Theta_E & c\Theta_E & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 \\ s\theta_{123} & c\theta_{123} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

OR $\rightarrow \Theta_E = \theta_1 + \theta_2 + \theta_3$

Forward Kinematics

- Successive Screw Method



  - ✓ Example 2
    - Consider 6DoF Elbow Manipulator (6R)
    - Reference Pose:

Consider all the joint variables $\theta_i = 0$.

Consider the base frame $x - y - z$ and

The end effector frame $u - v - w$

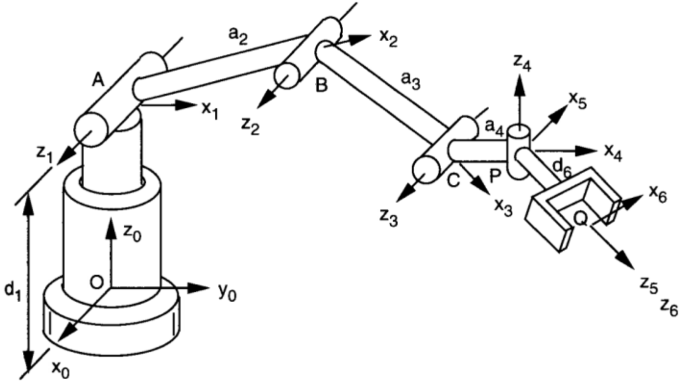
Derive the reference pose by:

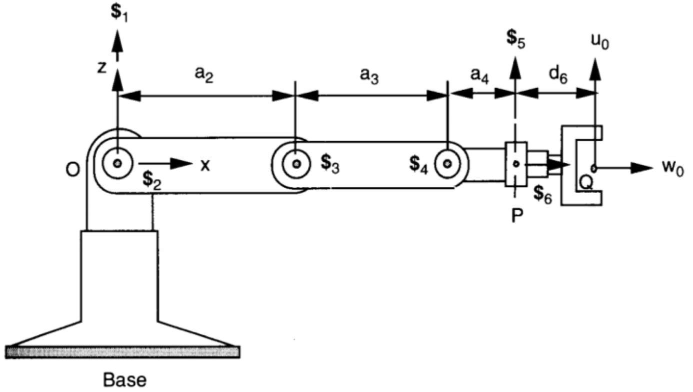
$$\boldsymbol{u}_0 = [0,0,1]^T, \boldsymbol{v}_0 = [0,-1,0]^T, \boldsymbol{w}_0 = [1,0,0]^T$$

$$\boldsymbol{x}_E^0 = \boldsymbol{q}_0 = [a_2 + a_3 + a_4 + d_6, 0,0]^T$$

And for the wrist point $\boldsymbol{P}$

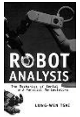
$$\boldsymbol{p}_0 = [a_2 + a_3 + a_4, 0,0]^T.$$





Forward Kinematics

- Successive Screw Method



✓ Example 2: (Cont.)

- Reference Pose:

  Denote the screws $\hat{\$}_i$ as on the diagram

  Find the screw parameters $\hat{\boldsymbol{s}}_i$ and $\boldsymbol{s}_{oi}$ as given in the table

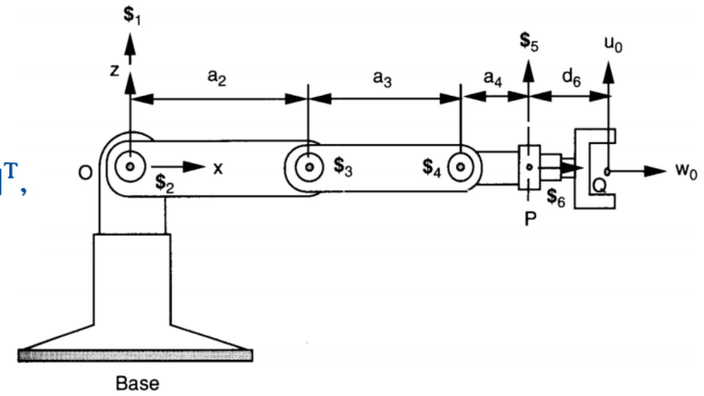
Joint i	$\mathbf{s}_i$	$\mathbf{s}_{oi}$
1	$(0, 0, 1)$	$(0, 0, 0)$
2	$(0, -1, 0)$	$(0, 0, 0)$
3	$(0, -1, 0)$	$(a_2, 0, 0)$
4	$(0, -1, 0)$	$(a_2 + a_3, 0, 0)$
5	$(0, 0, 1)$	$(a_2 + a_3 + a_4, 0, 0)$
6	$(1, 0, 0)$	$(0, 0, 0)$

- Target Pose:

  Let the target position of the wrist be:

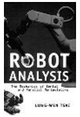
$$\mathbf{u} = [u_x, u_y, u_z]^{\mathrm{T}}, \quad \mathbf{v} = [v_x, v_y, v_z]^{\mathrm{T}}, \quad \mathbf{w} = [w_x, w_y, w_z]^{\mathrm{T}},$$
$$\mathbf{p} = [p_x, p_y, p_z]^{\mathrm{T}}.$$

  And for the end effector $\boldsymbol{q} = [q_x, q_y, q_z]^T$.



Forward Kinematics

- Successive Screw Method



  - ✓ Example 2: (Cont.)
    - Screw Transformation Matrices:

$$A_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} c\theta_2 & 0 & -s\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ s\theta_2 & 0 & c\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} c\theta_3 & 0 & -s\theta_3 & a_2(1 - c\theta_3) \\ 0 & 1 & 0 & 0 \\ s\theta_3 & 0 & c\theta_3 & -a_2 s\theta_3 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & (a_2 + a_3)(1 - c\theta_4) \\ 0 & 1 & 0 & 0 \\ s\theta_4 & 0 & c\theta_4 & -(a_2 + a_3)s\theta_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & (a_2 + a_3 + a_4)(1 - c\theta_5) \\ s\theta_5 & c\theta_5 & 0 & -(a_2 + a_3 + a_4)s\theta_5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta_6 & -s\theta_6 & 0 \\ 0 & s\theta_6 & c\theta_6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Forward Kinematics

- Successive Screw Method
  - ✓ Example 3
    - Consider 6DoF Stanford Arm (2RP3R)
    - Reference Pose:

Consider all the joint variables $\theta_i = d_i = 0$.

Consider the base frame $x - y - z$ and

The end effector frame $u - v - w$

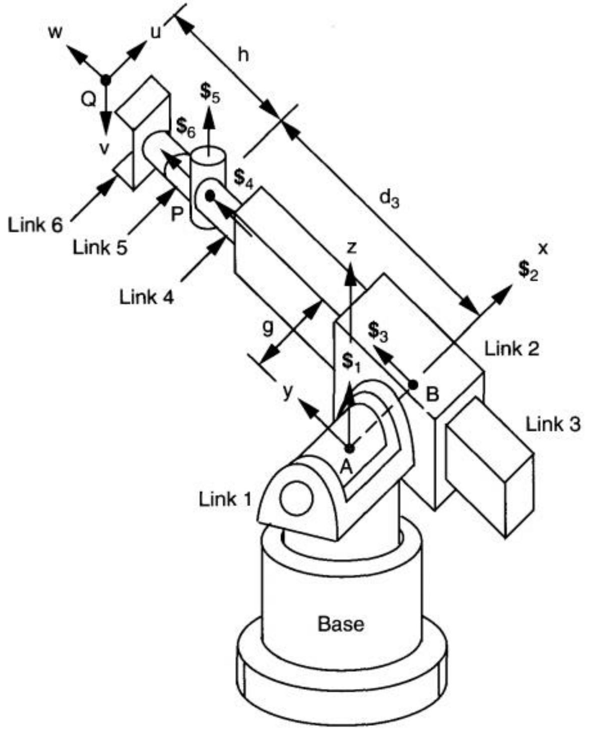
Derive the reference pose by:

$$\boldsymbol{u}_0 = [1, 0, 0]^T, \boldsymbol{v}_0 = [0,0,-1]^T, \boldsymbol{w}_0 = [0, 1, 0]^T$$

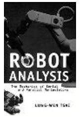
$$\boldsymbol{x}_E^0 = \boldsymbol{q}_0 = [g, h, 0]^T$$

And for the wrist point $\boldsymbol{P}$

$$\boldsymbol{p}_0 = [g, 0,0]^T.$$



Forward Kinematics



- Successive Screw Method
  - ✓ Example 3: (Cont.)
    - Reference Pose:

      Denote the screws $\hat{\$}_i$ as on the diagram

      Find the screw parameters $\hat{\boldsymbol{s}}_i$ and $\boldsymbol{s}_{\boldsymbol{o}_i}$ as given in the table

Joint i	$\mathbf{s}_i\,(s_x, s_y, s_z)$	$\mathbf{s}_{oi}\,(s_{ox}, s_{oy}, s_{oz})$
1	(0, 0, 1)	(0, 0, 0)
2	(1, 0, 0)	(0, 0, 0)
3	(0, 1, 0)	(g, 0, 0)
4	(0, 1, 0)	(g, 0, 0)
5	(0, 0, 1)	(g, 0, 0)
6	(0, 1, 0)	(g, 0, 0)

    - Target Pose:

      Let the target position of the wrist be:

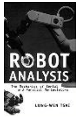
$$\mathbf{u} = [u_x, u_y, u_z]^{\mathrm{T}}, \quad \mathbf{v} = [v_x, v_y, v_z]^{\mathrm{T}}, \quad \mathbf{w} = [w_x, w_y, w_z]^{\mathrm{T}},$$
$$\mathbf{p} = [p_x, p_y, p_z]^{\mathrm{T}}.$$

      And for the end effector $\boldsymbol{q} = \left[q_x, q_y, q_z\right]^T$,

      Where $\boldsymbol{p} = \boldsymbol{q} - h\boldsymbol{w}$.

Forward Kinematics

- Successive Screw Method



✓ Example 3: (Cont.)

▪ Screw Transformation Matrices:

$$A_1A_2A_3 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta_2 & -s\theta_2 & 0 \\ 0 & s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

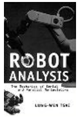
$$= \begin{bmatrix} c\theta_1 & -s\theta_1 c\theta_2 & s\theta_1 s\theta_2 & -d_3 s\theta_1 c\theta_2 \\ s\theta_1 & c\theta_1 c\theta_2 & -c\theta_1 s\theta_2 & d_3 c\theta_1 c\theta_2 \\ 0 & s\theta_2 & c\theta_2 & d_3 s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.165}$$

Wrist Position: $\boldsymbol{p} = A_1 A_2 A_3\, \boldsymbol{p_0}$

$$p_x = g c\theta_1 - d_3 s\theta_1 c\theta_2,$$
$$p_y = g s\theta_1 + d_3 c\theta_1 c\theta_2,$$
$$p_z = d_3 s\theta_2.$$

Forward Kinematics

- Successive Screw Method



✓ Example 3: (Cont.)

- Screw Transformation Matrices:

End Effector Orientation:

$$R_4 R_5 R_6 = \begin{bmatrix} c\theta_4 & 0 & s\theta_4 \\ 0 & 1 & 0 \\ -s\theta_4 & 0 & c\theta_4 \end{bmatrix} \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 \\ s\theta_5 & c\theta_5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_6 & 0 & s\theta_6 \\ 0 & 1 & 0 \\ -s\theta_6 & 0 & c\theta_6 \end{bmatrix}$$

$$= \begin{bmatrix} c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6 & -c\theta_4 s\theta_5 & c\theta_4 c\theta_5 s\theta_6 + s\theta_4 c\theta_6 \\ s\theta_5 c\theta_6 & c\theta_5 & s\theta_5 s\theta_6 \\ -s\theta_4 c\theta_5 c\theta_6 - c\theta_4 s\theta_6 & s\theta_4 s\theta_5 & -s\theta_4 c\theta_5 s\theta_6 + c\theta_4 c\theta_6 \end{bmatrix},$$

Forward Kinematics

- Frame Terminology
  - ✓ The Base Frame, $\{B\}$
  - ✓ The Station Frame, $\{S\}$
  - ✓ The Wrist Frame, $\{W\}$
  - ✓ The Tool Frame, $\{T\}$
  - ✓ The Goal Frame, $\{G\}$
    - Where is The Tool?

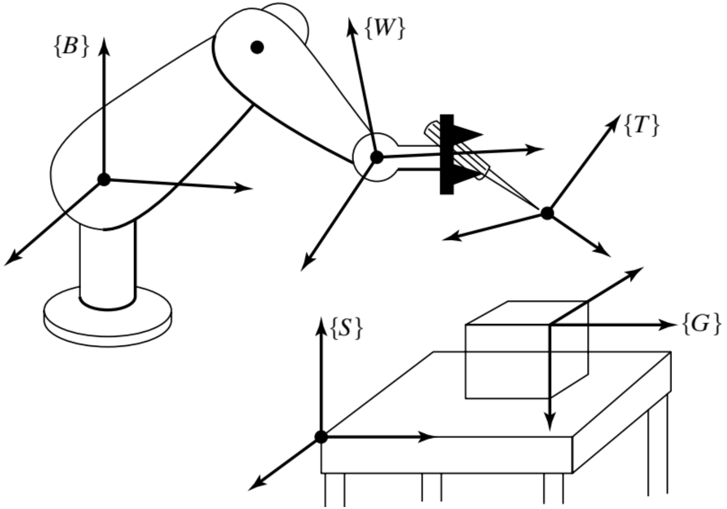
      Where means the position and orientation

      ${}^S_T T$ indicates where the tool is with respect to the station frame $\{S\}$

$$ {}^S_T T = {}^B_S T^{-1}\, {}^B_W T\, {}^W_T T. $$

To reach the tool to the goal one may solve the kinematics problem of:

$$ {}^S_T T = {}^S_G T. $$



Contents

1 Introduction

Definitions, kinematic loop closure, forward and inverse kinematics, joint and task space variables,

2 Forward Kinematics

Motivating example, geometric and algorithmic approach, frame assignment, DH parameters, Craig's and Paul's conventions, DH homogeneous transformations, case studies. Successive screw method., Screw-based transformations, Case studies, frame terminology.

3 Inverse Kinematics

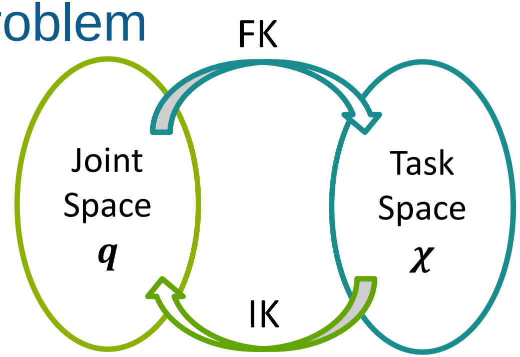
Inverse problem, solvability, existence of solutions, reachable and dexterous workspace. Methods of solution, Algebraic, trigonometric, geometric solutions, reduction to polynomials, Pieper's solution, method of successive screws, Case studies.

In this chapter we define the forward and inverse kinematic of serial manipulators. Different geometric, algorithmic, and screw-based solution methods will be examined, and Denavit- Hartenberg, and homogeneous transformation is introduced. The loop closure method in forward and inverse problem is solved for a number of case studies.

Kinematic Analysis

- Inverse Kinematics
  - ✓ Solve the inverse problem
    - Forward kinematics

      Given $\boldsymbol{q}$ find $\boldsymbol{\chi}$



Inverse kinematics

Given $\boldsymbol{\chi}$ find $\boldsymbol{q}$

- Solve the loop closure equation for: Given χ find $\boldsymbol{q}$

$$ {}^{0}_{E}T = {}^{0}_{\mathrm{n}}T = {}^{0}_{1}T \; {}^{1}_{2}T \cdots {}^{\mathrm{n-2}}_{\mathrm{n-1}}T \;\; {}^{\mathrm{n-1}}_{\mathrm{n}}T $$

$$ ({}^{2}_{3}T\ )^{-1}({}^{1}_{2}T\ )^{-1}({}^{0}_{1}T\ )^{-1}\,{}^{0}_{E}T = {}^{3}_{4}T \cdots {}^{\mathrm{n-2}}_{\mathrm{n-1}}T \;\; {}^{\mathrm{n-1}}_{\mathrm{n}}T $$

Direct inversion is not practical, and usually using the wrist frame $\{w\}$ for separation of position and orientation equation is very effective

Inverse Kinematic

- Solvability
  - ✓ The vector loop closure equations are nonlinear and trigonometric
    - For 6DoF manipulator, the number of variables are six
    - The number of equations are twelve (9 for ${}^{0}_{E}T$, 3 for x^{0}_{E})
    - Only three out of 9 equations are independent
    - Finding the solution is difficult
    - IK might have no solution (out of workspace) or multiple solutions
  - ✓ Existence of Solutions
    - Solution exists if the end-effector is within the reachable workspace of the robot (might have multiple solutions)
    - On the border of the reachable workspace the solution is unique.
    - Reachable Workspace (RW)

      The Volume of Space (6DoF space in general) that the end-effector of the robot can reach
    - Dexterous Workspace (DW)

      The Volume of space that the end-effector of the robot can reach with all configuration

      Dextereous Workspace is a subset of Reachable Workspace
    - Constant Orientation Workspace (COW)

      The Volume of Space (6DoF Space) that the end-effector of the robot can reach with constant configuration

      This is used to better visualization (6DoF workspace)

Inverse Kinematic

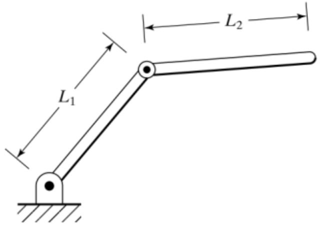
- Workspace
  - ✓ Reachable & Dexterous Workspace
    - Consider Planar RR Manipulator

IF $L_1 = L_2$ (with no joint limit)

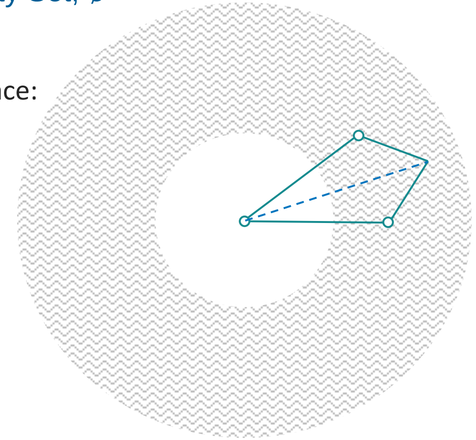
RW: A disc with radius $2L_1$ DW: Only one point; the origin

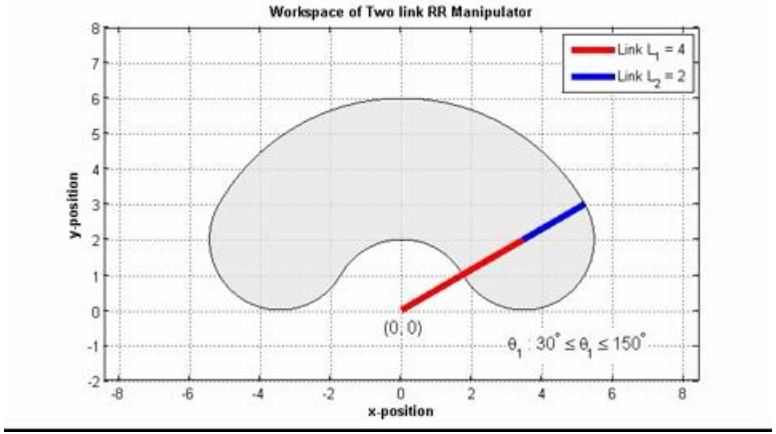
IF $L_1 > L_2$ (with no joint limit)

RW: A Ring with outer radius $L_1 + L_2$, and the inner radius $L_1 - L_2$;
DW: Empty Set; ∅



On the boundaries of the workspace:
One double solution for
Fully extended/folded Arm





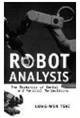
Inverse Kinematic

- Methods of Solution
  - ✓ Closed-Form Solution & Numerical Solution
    - Restrict ourselves to closed-form solution
    - IK of a 6DoF manipulator with R and P joints are solvable

Algebraic (Trigonometric) Solution
Analytical Geometric Solution
Reduction to Polynomial
Pieper’s Solution (When three axes intersects)
Method of Successive Screws

Inverse Kinematics

- Algebraic (Trigonometric) Solution
  - ✓ Kinematic loop closure
    - Consider Planar RRR Manipulator



Shorthand notation (FK)

$$a_1 c_1 + a_2 c_{12} + a_3 c_{123} = x_E$$
$$a_1 s_1 + a_2 s_{12} + a_3 s_{123} = y_E$$
$$\phi = \Theta_E = \theta_1 + \theta_2 + \theta_3$$

Consider the wrist point P vs the end effector point Q

$$p_x = x_E - a_3 c\phi; \quad p_y = y_E - a_3 s\phi$$

On the other side,

$$p_x = a_1 c\theta_1 + a_2 c\theta_{12},$$
$$p_y = a_1 s\theta_1 + a_2 s\theta_{12},$$

By this means θ_3 disappears. Note that the distance between P to O is independent of θ_1. Hence eliminate θ_1 by summing the squares of above equations:

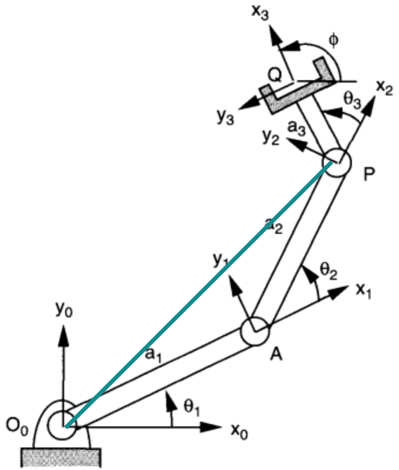
$$p_x^2 + p_y^2 = a_1^2 + a_2^2 + 2a_1 a_2 c\theta_2.$$

Solving for θ_2:

$$\theta_2 = \cos^{-1}\kappa,$$

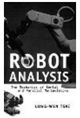
where

$$\kappa = \frac{p_x^2 + p_y^2 - a_1^2 - a_2^2}{2a_1 a_2}.$$



Inverse Kinematic

- Algebraic Approach
  - ✓ Kinematic loop closure



- Consider Planar RRR Manipulator

  Inverse cosine equation yields to two real roots if $|\kappa| < 1$.

  If $\theta_2 = \theta^*$ is the solution, then $\theta_2 = -\theta^*$ is as well.

  (Elbow up and Elbow down config)

  One double root if $|\kappa| = 1$ (Border of the workspace).

  No solution if $|\kappa| > 1$ (Out of workspace).

- Solve for θ_1 , by expanding the loop closure equations

$$(a_1 + a_2 c\theta_2)c\theta_1 - (a_2 s\theta_2)s\theta_1 = p_x,$$

$$(a_2 s\theta_2)c\theta_1 + (a_1 + a_2 c\theta_2)s\theta_1 = p_y.$$

Solve for θ_1

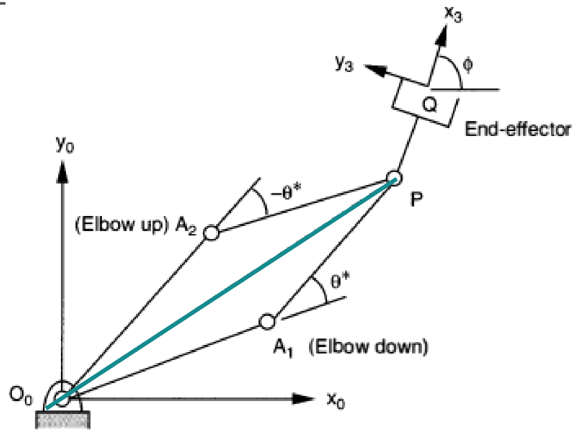
$$c\theta_1 = \frac{p_x(a_1 + a_2 c\theta_2) + p_y a_2 s\theta_2}{\Delta},$$

$$s\theta_1 = \frac{-p_x a_2 s\theta_2 + p_y(a_1 + a_2 c\theta_2)}{\Delta}.$$

In which

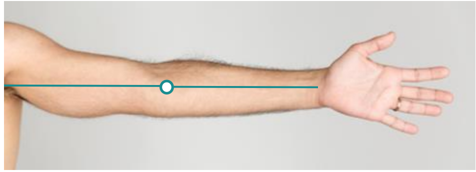
$$\Delta = a_1^2 + a_2^2 + 2a_1 a_2 c\theta_2.$$

We might use Atan2. $\theta_1 = \mathrm{Atan2}(s\theta_1, c\theta_1).$



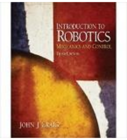
Two solutions for $|\kappa| < 1$

One double root for fully extended Arm



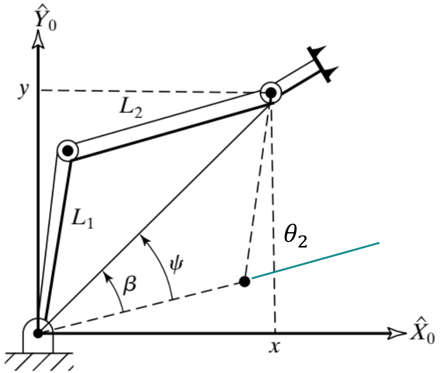
Inverse Kinematics

- Geometric Approach



✓ Decompose the spatial geometry to a number of plane-geometry

- When $\alpha_i = 0$ or $\pm\pi/2$ it is plausible.
- Consider Planar RRR Manipulator



Consider the solid triangle apply the law of cosines

$$x^2 + y^2 = l_1^2 + l_2^2 - 2l_1 l_2 \cos(180 - \theta_2).$$

In which, $\cos(180 - \theta_2) = -c_2$, therefore

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}.$$

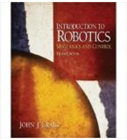
This has a solution if the radius of the goal point P is less than $l_1 + l_2$.

To solve for θ_1 follow the procedure given in the previous slide.

Note that the geometric approach gives much faster solution.

Inverse Kinematic

- Reduction to Polynomial



✓ Trigonometric relation to polynomial

  - Use half-angle tangent relation

$$u = \tan\frac{\theta}{2}, \qquad \cos\theta = \frac{1-u^2}{1+u^2}, \qquad \sin\theta = \frac{2u}{1+u^2}.$$

Example: Convert the following equation to algebraic polynomials

$$a\cos\theta + b\sin\theta = c$$

Substitute the above relation and manipulate:

$$a(1-u^2) + 2bu = c(1+u^2).$$

Collecting the powers:

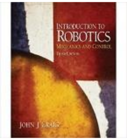
$$(a+c)u^2 - 2bu + (c-a) = 0,$$

This is solved by quadratic polynomial solution $u = \dfrac{b \pm \sqrt{b^2+a^2-c^2}}{a+c}.$

Hence: $$\theta = 2\tan^{-1}\left(\frac{b \pm \sqrt{b^2+a^2-c^2}}{a+c}\right).$$

Inverse Kinematic

- Pieper’s Solution (When three axes intersects)



  ✓ For 6DoF robots with three consecutive joint axis intersection

    ▪ Consider the three last joint intersecting (most of commercial Robots)

Origin of frames $\{4\}, \{5\}$, and $\{6\}$ are at the point of intersection

$$P_{O4}^{0} = {}_{1}^{0}T \, {}_{2}^{1}T \, {}_{3}^{2}T \, P_{O4}^{3}$$

Use DH-convention

$$P_{O4}^{0} = {}_{1}^{0}T \, {}_{2}^{1}T \, {}_{3}^{2}T \begin{bmatrix} a_3 \\ -d_4 s\alpha_3 \\ d_4 c\alpha_3 \\ 1 \end{bmatrix}$$

Determine this point by calculation of general DH-transformation ${}_{1}^{0}T \, {}_{2}^{1}T \, {}_{3}^{2}T$

$$P_{O4}^{0} = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix}$$

In which

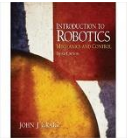
$$g_1 = c_2 f_1 - s_2 f_2 + a_1,$$
$$g_2 = s_2 c\alpha_1 f_1 + c_2 c\alpha_1 f_2 - s\alpha_1 f_3 - d_2 s\alpha_1,$$
$$g_3 = s_2 s\alpha_1 f_1 + c_2 s\alpha_1 f_2 + c\alpha_1 f_3 + d_2 c\alpha_1.$$

while,

$$f_1 = a_3 c_3 + d_4 s\alpha_3 s_3 + a_2,$$
$$f_2 = a_3 c\alpha_2 s_3 - d_4 s\alpha_3 c\alpha_2 c_3 - d_4 s\alpha_2 c\alpha_3 - d_3 s\alpha_2,$$
$$f_3 = a_3 s\alpha_2 s_3 - d_4 s\alpha_3 s\alpha_2 c_3 + d_4 c\alpha_2 c\alpha_3 + d_3 c\alpha_2.$$

Inverse Kinematic

- Pieper’s Solution (When three axes intersects)



  - Find magnitude of P_{O4}^{0} as r

$r = g_1^2 + g_2^2 + g_3^2;$ therefore $r = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 + 2a_1(c_2 f_1 - s_2 f_2).$

Write this equation along with the z-component of P_{O4}^{0}

$$r = (k_1 c_2 + k_2 s_2) 2a_1 + k_3,$$
$$z = (k_1 s_2 - k_2 c_2) s\alpha_1 + k_4,$$

where

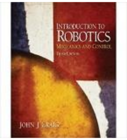
$$k_1 = f_1,$$
$$k_2 = -f_2,$$
$$k_3 = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3,$$
$$k_4 = f_3 c\alpha_1 + d_2 c\alpha_1.$$

In this equation θ_1 is eliminated and is of simple form of θ_2. Now Solve for θ_3:

1. If $a_1 = 0$, then we have $r = k_3$, where r is known. The right-hand side (k_3) is a function of θ_3 only. After the substitution (4.35), a quadratic equation in $\tan\frac{\theta_3}{2}$ may be solved for θ_3.
2. If $s\alpha_1 = 0$, then we have $z = k_4$, where z is known. Again, after substituting via (4.35), a quadratic equation arises that can be solved for θ_3.

Inverse Kinematic

- Pieper's Solution (When three axes intersects)



Furthermore,

3. Otherwise, eliminate s_2 and c_2 from (4.50) to obtain

$$\frac{(r-k_3)^2}{4a_1^2} + \frac{(z-k_4)^2}{s^2\alpha_1} = k_1^2 + k_2^2. \qquad (4.52)$$

This equation, after the (4.35) substitution for θ_3, results in an equation of degree 4, which can be solved for θ_3.[3]

Having solved for θ_3 , one may solve for θ_2, and then for θ_1.

- Complete the solution:

We need to solve for $\theta_4, \theta_5, \theta_6$, since these axis intersect:

These joint angles determines the orientation of the end effector

This can be computed by the orientation of the goal ${}^0_6R$.

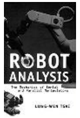
First determine the orientation of link frame $\{4\}$ relative to the base frame when $\theta_4 = 0$, denote this with ${}^0_4R|_{\theta_4=0}$. This found by

$${}^4_6R|_{\theta_4=0} = {}^0_4R^{-1}|_{\theta_4=0}\, {}^0_6R.$$

This part of the orientation may be found by suitable Euler angles, usually $Z-Y-Z$ one.

Inverse Kinematic

- Examples:
  - ✓ See IK solution examples in the references.



  - Example 1: Consider the 6DoF Fanuc S-900w robot
  - Algorithmic Approach (Paul's Convention)

Denote $\boldsymbol{q} = [\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6]^T$ and $\boldsymbol{\chi} = [\boldsymbol{x}_E, \Theta_E]^T$

Affix the frames and find DH-parameters as given in the next slide.

Find the homogeneous transformations:

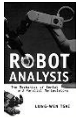
$$
{}^0A_1 = \begin{bmatrix} c\theta_1 & 0 & s\theta_1 & a_1 c\theta_1 \\ s\theta_1 & 0 & -c\theta_1 & a_1 s\theta_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^1A_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & a_2 c\theta_2 \\ s\theta_2 & c\theta_2 & 0 & a_2 s\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^2A_3 = \begin{bmatrix} c\theta_3 & 0 & s\theta_3 & a_3 c\theta_3 \\ s\theta_3 & 0 & -c\theta_3 & a_3 s\theta_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
$$

$$
{}^3A_4 = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & 0 \\ s\theta_4 & 0 & c\theta_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^4A_5 = \begin{bmatrix} c\theta_5 & 0 & s\theta_5 & 0 \\ s\theta_5 & 0 & -c\theta_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad {}^5A_6 = \begin{bmatrix} c\theta_6 & -s\theta_6 & 0 & 0 \\ s\theta_6 & c\theta_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

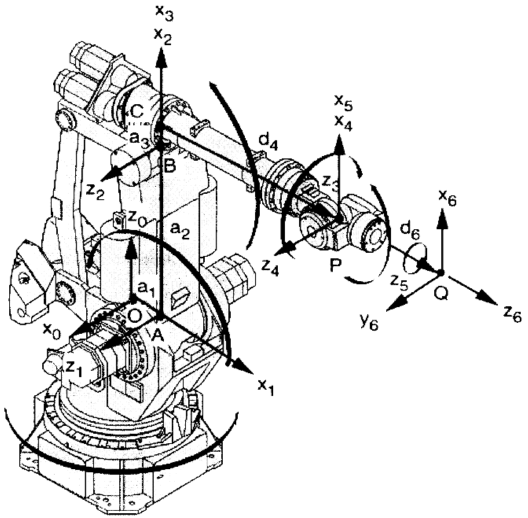
$$
{}^0A_6 = \begin{bmatrix} u_x & v_x & w_x & q_x \\ u_y & v_y & w_y & q_y \\ u_z & v_z & w_z & q_z \\ 0 & 0 & 0 & 1 \end{bmatrix}.
$$

Inverse Kinematic

- Example 1: Fanuc S-900w robot



Algorithmic Approach (Paul’s Convention)



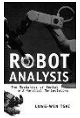
Denavit-Hartenberg parameters

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	a_1	0	θ_1
2	0	a_2	0	θ_2
3	$\pi/2$	a_3	0	θ_3
4	$-\pi/2$	0	d_4	θ_4
5	$\pi/2$	0	0	θ_5
6	0	0	d_6	θ_6

Inverse Kinematic

- Example 1: Fanuc S-900w robot

  Note that the last three joints intersect @ $\boldsymbol{P}$



While: $^6\mathbf{p} = \overline{QP} = [0,0,-d_6,1]^T.$ and: $^0\mathbf{p} = \overline{OP} = \begin{bmatrix} p_x \\ p_x \\ p_z \\ 1 \end{bmatrix} = \begin{bmatrix} q_x - d_6 w_x \\ q_x - d_6 w_y \\ q_z - d_6 w_z \\ 1 \end{bmatrix}.$

By inspection it is easy to see

$$^3\mathbf{p} = \overline{CP} = [0,0,d_4,1]^T.$$

Transform it to the base frame by: $^0\mathbf{p} = {}^0A_3\,{}^3\mathbf{p}.$

Now manipulate $({}^0A_1)^{-1}\,{}^0\mathbf{p} = {}^1A_3\,{}^3\mathbf{p}.$

Use homogeneous transformations to find …

$$p_x c\theta_1 + p_y s\theta_1 - a_1 = a_2 c\theta_2 + a_3 c\theta_{23} + d_4 s\theta_{23}, \qquad (2.81)$$

$$p_z = a_2 s\theta_2 + a_3 s\theta_{23} - d_4 c\theta_{23}, \qquad (2.82)$$

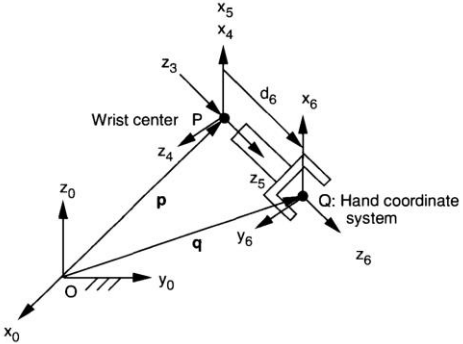
$$p_x s\theta_1 - p_y c\theta_1 = 0, \qquad (2.83)$$

Where p_x, p_y, p_z is found from q_x, q_y, q_z.

A solution for θ_1 is found by the last equation:

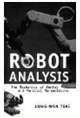
$$\theta_1 = \tan^{-1}\frac{p_y}{p_x}.$$

There will be two solutions for this joint angle.



Inverse Kinematic

- Example 1: Fanuc S-900w robot



By observation the distance between $\boldsymbol{A}$ and $\boldsymbol{P}$ is independent to θ_1 and θ_2. Therefore these two variables can be eliminated simultaneously

Sum the squares of (2.81) – (2.83)

$$\kappa_1 s\theta_3 + \kappa_2 c\theta_3 = \kappa_3, \tag{2.85}$$

Where
$$\kappa_1 = 2a_2 d_4,\ \kappa_2 = 2a_2 a_3,$$
$$\kappa_3 = p_x^2 + p_y^2 + p_z^2 - 2p_x a_1 c\theta_1 - 2p_y a_1 s\theta_1 + a_1^2 - a_2^2 - a_3^2 - d_4^2.$$

Convert (2.85) into polynomial by half angle relations

$$c\theta_3 = \frac{1 - t_3^2}{1 + t_3^2} \quad \text{and} \quad s\theta_3 = \frac{2t_3}{1 + t_3^2}, \quad \text{where} \quad t_3 = \tan\frac{\theta_3}{2}.$$

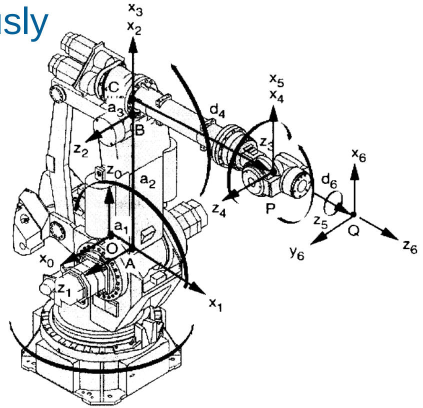
to obtain

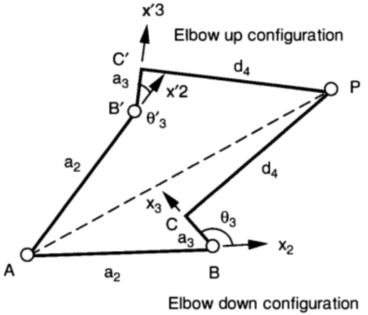
$$(\kappa_3 + \kappa_2)t_3^2 - 2\kappa_1 t_3 + (\kappa_3 - \kappa_2) = 0. \tag{2.86}$$

Hence,

$$\frac{\theta_3}{2} = \tan^{-1}\frac{\kappa_1 \pm \sqrt{\kappa_1^2 + \kappa_2^2 - \kappa_3^2}}{\kappa_3 + \kappa_2}. \tag{2.87}$$

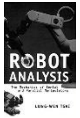
This yields to two real roots (Elbow up and down solution)





Inverse Kinematic

- Example 1: Fanuc S-900w robot



Once θ_1 and θ_3 are known, θ_2 is found by back substitution

Expand (2.81) and (2.82)

$$\mu_1 c\theta_2 + \nu_1 s\theta_2 = \gamma_1, \tag{2.88}$$

$$\mu_2 c\theta_2 + \nu_2 s\theta_2 = \gamma_2, \tag{2.89}$$

Where

$$\mu_1 = a_2 + a_3 c\theta_3 + d_4 s\theta_3, \qquad \mu_2 = a_3 s\theta_3 - d_4 c\theta_3,$$
$$\nu_1 = -a_3 s\theta_3 + d_4 c\theta_3, \qquad \nu_2 = a_2 + a_3 c\theta_3 + d_4 s\theta_3,$$
$$\gamma_1 = p_x c\theta_1 + p_y s\theta_1 - a_1, \qquad \gamma_2 = p_z.$$

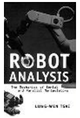
Solve (2.88) and (2.89) for $c\theta_2$ and $c\theta_2$, a unique solution is found for θ_2

$$\theta_2 = \mathrm{Atan2}(s\theta_2, c\theta_2). \tag{2.90}$$

Four solution is found for wrist position, but only two is physically possible.

Inverse Kinematic

- Example 1: Fanuc S-900w robot



Loop closure equation for end effector orientation:

$$^{3}A_{6} = (^{0}A_{3})^{-1}\, {}^{0}A_{6}. \qquad (2.91)$$

In which $({}^{0}_{3}A)^{-1}$ can be found, knowing θ_1, θ_2 and θ_3

Using homogeneous transformations

$$^{0}A_{3} = {}^{0}A_{1}\, {}^{1}A_{2}\, {}^{2}A_{3} = \begin{bmatrix} c\theta_1 c\theta_{23} & s\theta_1 & c\theta_1 s\theta_{23} & c\theta_1 (a_1 + a_2 c\theta_2 + a_3 c\theta_{23}) \\ s\theta_1 c\theta_{23} & -c\theta_1 & s\theta_1 s\theta_{23} & s\theta_1 (a_1 + a_2 c\theta_2 + a_3 c\theta_{23}) \\ s\theta_{23} & 0 & -c\theta_{23} & a_2 s\theta_2 + a_3 s\theta_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$^{3}A_{6} = {}^{3}A_{4}\, {}^{4}A_{5}\, {}^{5}A_{6} = \begin{bmatrix} c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6 & -c\theta_4 c\theta_5 s\theta_6 - s\theta_4 c\theta_6 & c\theta_4 s\theta_5 & d_6 c\theta_4 s\theta_5 \\ s\theta_4 c\theta_5 c\theta_6 + c\theta_4 s\theta_6 & -s\theta_4 c\theta_5 s\theta_6 + c\theta_4 c\theta_6 & s\theta_4 s\theta_5 & d_6 s\theta_4 s\theta_5 \\ -s\theta_5 c\theta_6 & s\theta_5 s\theta_6 & c\theta_5 & d_4 + d_6 c\theta_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

From the rotation matrices θ_5 can be found

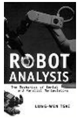
$$\theta_5 = \cos^{-1} r_{33}, \qquad (2.92)$$

$$r_{33} = w_x c\theta_1 s\theta_{23} + w_y s\theta_1 s\theta_{23} - w_z c\theta_{23}.$$

Two real roots are found.

Inverse Kinematic

- Example 1: Fanuc S-900w robot



Assuming $s\theta_1 \neq 0$ we can solve for θ_4 and θ_6

Use $(1{,}3)$ and $(2{,}3)$ components of the rotation matrix:

$$c\theta_4 = \frac{w_x c\theta_1 c\theta_{23} + w_y s\theta_1 c\theta_{23} + w_z s\theta_{23}}{s\theta_5}. \qquad s\theta_4 = \frac{w_x s\theta_1 - w_y c\theta_1}{s\theta_5}.$$

Hence, a unique solution can be found by

$$\theta_4 = \mathrm{Atan2}(s\theta_4, c\theta_4).$$

Similarly, se $(3{,}1)$ and $(3{,}2)$ components of the rotation matrix:

$$c\theta_6 = -\frac{u_x c\theta_1 s\theta_{23} + u_y s\theta_1 s\theta_{23} - u_z c\theta_{23}}{s\theta_5}.$$

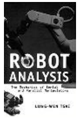
$$s\theta_6 = \frac{v_x c\theta_1 s\theta_{23} + v_y s\theta_1 s\theta_{23} - v_z c\theta_{23}}{s\theta_5}.$$

A unique solution of θ_6 is found

$$\theta_6 = \mathrm{Atan2}(s\theta_6, c\theta_6).$$

Inverse Kinematic

- Method of Successive Screws



✓ Loop closure for wrist position $\boldsymbol{P}$

$$\boldsymbol{P} = A_1\, A_2 \ldots A_i\, \boldsymbol{P_0}$$

- Matrix manipulation: for given $\boldsymbol{x}$, find the joint space variable $\boldsymbol{q}$.

$$A_1^{-1}\boldsymbol{P} = A_2 \ldots A_i\, \boldsymbol{P_0}\ , \ \ldots$$

✓ Loop closure end effector orientation.

use $\quad \boldsymbol{w} = R_1\, R_2 \ldots R_6\, \boldsymbol{w_0}$

or $\quad \boldsymbol{u} = R_1\, R_2 \ldots R_6\, \boldsymbol{u_0}$

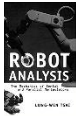
or $\quad \boldsymbol{v} = R_1\, R_2 \ldots R_6\, \boldsymbol{v_0}$

- Where R_i is the rotation matrix corresponding to A_i
- Matrix manipulation: for given $\boldsymbol{x}$, find the joint space variable $\boldsymbol{q}$.

$$R_3^T R_2^T R_1^T \boldsymbol{w} = R_4\, R_5 R_6\, \boldsymbol{w_0},\ \ldots;$$

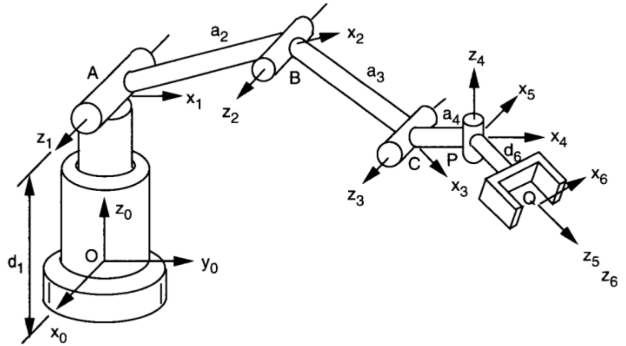
Inverse Kinematics

- Successive Screw Method



✓ Example 1

- Consider 6DoF Elbow Manipulator (6R)



Screw Transformation Matrices:

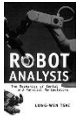
$$A_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} c\theta_2 & 0 & -s\theta_2 & 0 \\ 0 & 1 & 0 & 0 \\ s\theta_2 & 0 & c\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} c\theta_3 & 0 & -s\theta_3 & a_2(1-c\theta_3) \\ 0 & 1 & 0 & 0 \\ s\theta_3 & 0 & c\theta_3 & -a_2 s\theta_3 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} c\theta_4 & 0 & -s\theta_4 & (a_2+a_3)(1-c\theta_4) \\ 0 & 1 & 0 & 0 \\ s\theta_4 & 0 & c\theta_4 & -(a_2+a_3)s\theta_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 & (a_2+a_3+a_4)(1-c\theta_5) \\ s\theta_5 & c\theta_5 & 0 & -(a_2+a_3+a_4)s\theta_5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta_6 & -s\theta_6 & 0 \\ 0 & s\theta_6 & c\theta_6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Inverse Kinematics

- Successive Screw Method



  - ✓ Example 1: (Cont.)
  - ✓ Loop closure for wrist position ***P***

$$\boldsymbol{P} = A_1 A_2 A_3 A_4\, \boldsymbol{P_0}$$

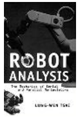
Manipulate

$$A_1^{-1}\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} = A_2 A_3 A_4 \begin{bmatrix} a_2 + a_3 + a_4 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad ({}^0A_1)^{-1} = \begin{bmatrix} c\theta_1 & s\theta_1 & 0 & 0 \\ -s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A_2 A_3 A_4 = \begin{bmatrix} c\theta_{234} & 0 & -s\theta_{234} & a_2 c\theta_2 + a_3 c\theta_{23} - (a_2 + a_3) c\theta_{234} \\ 0 & 1 & 0 & 0 \\ s\theta_{234} & 0 & c\theta_{234} & a_2 s\theta_2 + a_3 s\theta_{23} - (a_2 + a_3) s\theta_{234} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

Inverse Kinematics

- Successive Screw Method



✓ Example 1: (Cont.)

- This leads to:

$$p_x c\theta_1 + p_y s\theta_1 = a_2 c\theta_2 + a_3 c\theta_{23} + a_4 c\theta_{234}, \quad (2.144)$$

$$-p_x s\theta_1 + p_y c\theta_1 = 0, \quad (2.145)$$

$$p_z = a_2 s\theta_2 + a_3 s\theta_{23} + a_4 s\theta_{234}. \quad (2.146)$$

From (2.145) two solutions are found by

$$\theta_1 = \tan^{-1}\frac{p_y}{p_x}. \quad (2.147)$$

- For this manipulator position and orientation is not decoupled

Write the orientation loop closure

$$R_1^{\mathrm{T}}\mathbf{w} = R_2 R_3 R_4 R_5 \mathbf{w}_0, \quad (2.148)$$

This results in:

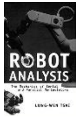
$$w_x c\theta_1 + w_y s\theta_1 = c\theta_{234} c\theta_5, \quad (2.149)$$

$$-w_x s\theta_1 + w_y c\theta_1 = s\theta_5, \quad (2.150)$$

$$w_z = s\theta_{234} c\theta_5. \quad (2.151)$$

Inverse Kinematics

- Successive Screw Method



✓ Example 1: (Cont.)

From (2.150) two solutions are found by

$$\theta_5 = \sin^{-1}(-w_x s\theta_1 + w_y c\theta_1). \qquad (2.152)$$

Equations (2.149) and (2.151) may be used to solve for θ_{234}

$$\theta_{234} = \mathrm{Atan2}\left[w_z/c\theta_5, (w_x c\theta_1 + w_y s\theta_1)/c\theta_5\right]. \qquad (2.153)$$

Next use (2.144) and (2.146) to solve for θ_2 and θ_3. Lets rewrite them as

$$a_2 c\theta_2 + a_3 c\theta_{23} = k_1, \qquad (2.154)$$

$$a_2 s\theta_2 + a_3 s\theta_{23} = k_2, \qquad (2.155)$$

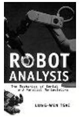
Where $k_1 = p_x c\theta_1 + p_y s\theta_1 - a_4 c\theta_{234}$ and $k_2 = p_z - a_4 s\theta_{234}$.

Summing the squares $$a_2^2 + a_3^2 + 2a_2 a_3 c\theta_3 = k_1^2 + k_2^2. \qquad (2.156)$$

This results in: $$\theta_3 = \cos^{-1}\frac{k_1^2 + k_2^2 - a_2^2 - a_3^2}{2a_2 a_3}. \qquad (2.157)$$

Inverse Kinematics

- Successive Screw Method



✓ Example 1: (Cont.)

To solve for θ_6, write the orientation loop closure for $\boldsymbol{u}$.

$$(R_1 R_2 R_3 R_4)^{\mathrm{T}} \mathbf{u} = R_5 R_6 \mathbf{u}_0. \qquad (2.158)$$

This results in:

$$u_x c\theta_1 c\theta_{234} + u_y s\theta_1 c\theta_{234} + u_z s\theta_{234} = s\theta_5 s\theta_6, \qquad (2.159)$$

$$-u_x s\theta_1 + u_y c\theta_1 = -c\theta_5 s\theta_6, \qquad (2.160)$$

$$-u_x c\theta_1 s\theta_{234} - u_y s\theta_1 s\theta_{234} + u_z c\theta_{234} = c\theta_6. \qquad (2.161)$$

Solve (2.159) and (2.160) for $s\theta_6$

$$s\theta_6 = s\theta_5 (u_x c\theta_1 c\theta_{234} + u_y s\theta_1 c\theta_{234} + u_z s\theta_{234}) - c\theta_5 (-u_x s\theta_1 + u_y c\theta_1). \quad (2.162)$$

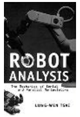
And use (2.161)

$$\theta_6 = \mathrm{Atan2}(s\theta_6, c\theta_6). \qquad (2.163)$$

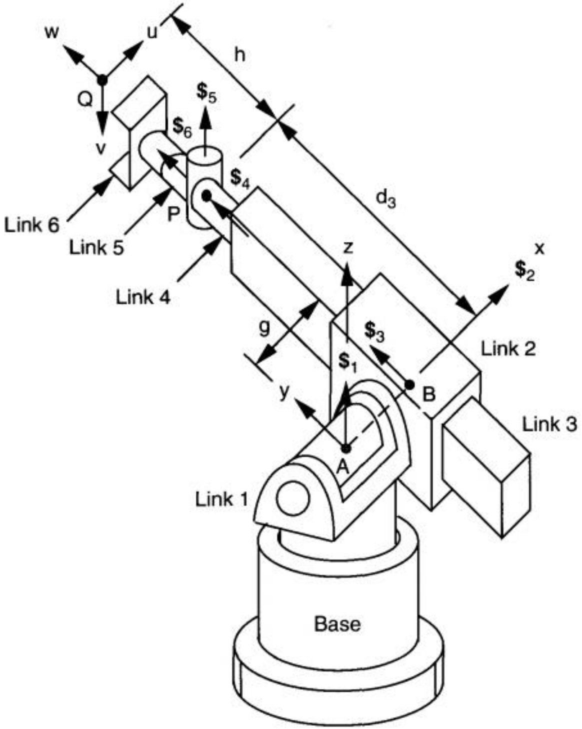
By this means the inverse kinematics is completed.

Inverse Kinematics

- Successive Screw Method



  - ✓ Example 2: Stanford Arm (2RP3R)
    - Screw Transformation Matrices:



$$A_1A_2A_3 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\theta_2 & -s\theta_2 & 0 \\ 0 & s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c\theta_1 & -s\theta_1 c\theta_2 & s\theta_1 s\theta_2 & -d_3 s\theta_1 c\theta_2 \\ s\theta_1 & c\theta_1 c\theta_2 & -c\theta_1 s\theta_2 & d_3 c\theta_1 c\theta_2 \\ 0 & s\theta_2 & c\theta_2 & d_3 s\theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (2.165)$$

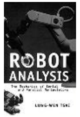
Wrist Position: $\boldsymbol{p} = A_1 A_2 A_3\, \boldsymbol{p_0}$

$$p_x = gc\theta_1 - d_3 s\theta_1 c\theta_2, \qquad (2.167)$$

$$p_y = gs\theta_1 + d_3 c\theta_1 c\theta_2, \qquad (2.168)$$

$$p_z = d_3 s\theta_2. \qquad (2.169)$$

- Successive Screw Method



✓ Example 2: (Cont.)

Find d_3 by summing the squares of the equations:

$$p_x^2 + p_y^2 + p_z^2 = g^2 + d_3^2. \qquad (2.170)$$

Hence,

$$d_3 = \pm\sqrt{p_x^2 + p_y^2 + p_z^2 - g^2}. \qquad (2.171)$$

This equation yield to two real roots if the end-effector is in the reachable workspace, but only the positive d_3 is acceptable.

Now use (2.169) to solve for θ_2:

$$\theta_2 = \sin^{-1}\frac{p_z}{d_3}. \qquad (2.172)$$

Here two solutions could be found, $\theta_2 = \theta_2^*, \pi - \theta_2^*$

Now solve for θ_1:

$$c\theta_1 = \frac{g p_x + d_3 p_y c\theta_2}{g^2 + d_3^2 c^2\theta_2}, \qquad (2.173)$$

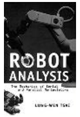
$$s\theta_1 = \frac{g p_y - d_3 p_x c\theta_2}{g^2 + d_3^2 c^2\theta_2}. \qquad (2.174)$$

hence

$$\theta_1 = \text{Atan2}(s\theta_1, c\theta_1). \qquad (2.175)$$

Inverse Kinematics

- Successive Screw Method



✓ Example 2: (Cont.)

- End effector orientation loop closure equation for $\boldsymbol{w}$

$$\mathbf{w} = R_1 R_2 R_3 R_4 R_5 R_6 \mathbf{w}_0. \tag{2.177}$$

Manipulate

$$R_3^T R_2^T R_1^T \mathbf{w} = R_4 R_5 R_6 \mathbf{w}_0. \tag{2.178}$$

In which

$$R_4 R_5 R_6 = \begin{bmatrix} c\theta_4 & 0 & s\theta_4 \\ 0 & 1 & 0 \\ -s\theta_4 & 0 & c\theta_4 \end{bmatrix} \begin{bmatrix} c\theta_5 & -s\theta_5 & 0 \\ s\theta_5 & c\theta_5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_6 & 0 & s\theta_6 \\ 0 & 1 & 0 \\ -s\theta_6 & 0 & c\theta_6 \end{bmatrix}$$

$$= \begin{bmatrix} c\theta_4 c\theta_5 c\theta_6 - s\theta_4 s\theta_6 & -c\theta_4 s\theta_5 & c\theta_4 c\theta_5 s\theta_6 + s\theta_4 c\theta_6 \\ s\theta_5 c\theta_6 & c\theta_5 & s\theta_5 s\theta_6 \\ -s\theta_4 c\theta_5 c\theta_6 - c\theta_4 s\theta_6 & s\theta_4 s\theta_5 & -s\theta_4 c\theta_5 s\theta_6 + c\theta_4 c\theta_6 \end{bmatrix},$$

Let us define ${}^3\mathbf{w} \equiv R_3^T R_2^T R_1^T \mathbf{w}$.

$$-c\theta_4 s\theta_5 = {}^3w_x, \tag{2.179}$$

$$c\theta_5 = {}^3w_y, \tag{2.180}$$

$$s\theta_4 s\theta_5 = {}^3w_z, \tag{2.181}$$

where

$${}^3w_x = w_x c\theta_1 + w_y s\theta_1,$$

$${}^3w_y = (-w_x s\theta_1 + w_y c\theta_1)c\theta_2 + w_z s\theta_2,$$

$${}^3w_z = (w_x s\theta_1 - w_y c\theta_1)s\theta_2 + w_z c\theta_2.$$

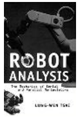
Note that $\boldsymbol{w}$ is independent of θ_6. Solve (2.180) for θ_5:

$$\theta_5 = \cos^{-1}({}^3w_y). \tag{2.182}$$

This equation yields to two real roots in the reachable workspace.

Inverse Kinematics

- Successive Screw Method



✓ Example 2: (Cont.)

Assume that $s\theta_5 \neq 0$ use (2.179) and (2.181) solve for θ_5:

$$\theta_4 = \mathrm{Atan2}({}^3w_z/\mathrm{s}\theta_5, -{}^3w_x/\mathrm{s}\theta_5). \tag{2.183}$$

Now solve for θ_6 by using loop closure equation for $\boldsymbol{u}$.

$$\mathbf{u} = R_1R_2R_3R_4R_5R_6\mathbf{u}_0. \tag{2.184}$$

Manipulate

$$R_3^{\mathrm{T}}R_2^{\mathrm{T}}R_1^{\mathrm{T}}\mathbf{u} = R_4R_5R_6\mathbf{u}_0. \tag{2.185}$$

Define ${}^3\mathbf{u} \equiv R_3^{\mathrm{T}}R_2^{\mathrm{T}}R_1^{\mathrm{T}}\mathbf{u}.$ This yields to:

$$
\begin{aligned}
{}^3u_x &= u_x\mathrm{c}\theta_1 + u_y\mathrm{s}\theta_1,\\
{}^3u_y &= (-u_x\mathrm{s}\theta_1 + u_y\mathrm{c}\theta_1)\mathrm{c}\theta_2 + u_z\mathrm{s}\theta_2,\\
{}^3u_z &= (u_x\mathrm{s}\theta_1 - u_y\mathrm{c}\theta_1)\mathrm{s}\theta_2 + u_z\mathrm{c}\theta_2.
\end{aligned}
$$

where

$$
\begin{aligned}
{}^3u_x &= \mathrm{c}\theta_4\mathrm{c}\theta_5\mathrm{c}\theta_6 - \mathrm{s}\theta_4\mathrm{s}\theta_6, && (2.186)\\
{}^3u_y &= \mathrm{s}\theta_5\mathrm{c}\theta_6, && (2.187)\\
{}^3u_z &= -\mathrm{s}\theta_4\mathrm{c}\theta_5\mathrm{c}\theta_6 - \mathrm{c}\theta_4\mathrm{s}\theta_6, && (2.188)
\end{aligned}
$$

Multiply (2.186) by $s\theta_4$ and (2.188) by $c\theta_4$

$$\mathrm{s}\theta_4\,{}^3u_x + \mathrm{c}\theta_4\,{}^3u_z = -\mathrm{s}\theta_6. \tag{2.189}$$

This yields to

$$\theta_6 = \mathrm{Atan2}(-\mathrm{s}\theta_4\,{}^3u_x - \mathrm{c}\theta_4\,{}^3u_z,\ {}^3u_y/\mathrm{s}\theta_5). \tag{2.190}$$

And completes the IK solution.

Hamid D. Taghirad
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About Hamid D. Taghirad

Hamid D. Taghirad has received his B.Sc. degree in mechanical engineering from Sharif University of Technology, Tehran, Iran, in 1989, his M.Sc. in mechanical engineering in 1993, and his Ph.D. in electrical engineering in 1997, both from McGill University, Montreal, Canada. He is currently the University Vice-Chancellor for Global strategies and International Affairs, Professor and the Director of the Advanced Robotics and Automated System (ARAS), Department of Systems and Control, Faculty of Electrical Engineering, K. N. Toosi University of Technology, Tehran, Iran. He is a senior member of IEEE, and Editorial board of International Journal of Robotics: Theory and Application, and International Journal of Advanced Robotic Systems. His research interest is *robust* and *nonlinear control* applied to *robotic systems*. His publications include five books, and more than 250 papers in international Journals and conference proceedings.

Robotics: Mechanics & Control

Chapter 3: Kinematic Analysis

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Thank You