Robotics: Mechanics & Control

Chapter 5: Dynamic Analysis

In this chapter we review the dynamics analysis for serial robots. First the definition to angular and linear accelerations are given, then mass properties, linear and angular momentums, and kinetic energy of a rigid body in space is defined. Lagrange formulation is given in general form, and dynamics mass matrix, gravity and Coriolis and centrifugal vectors are defined and derived for several case studies. Dynamic formulation properties is given next, then actuator dynamics is elaborated for electrically driven robots with gearbox. Finally, linear regression method is used for dynamics calibration, and model verification methods are elaborated by introducing consistency measure.

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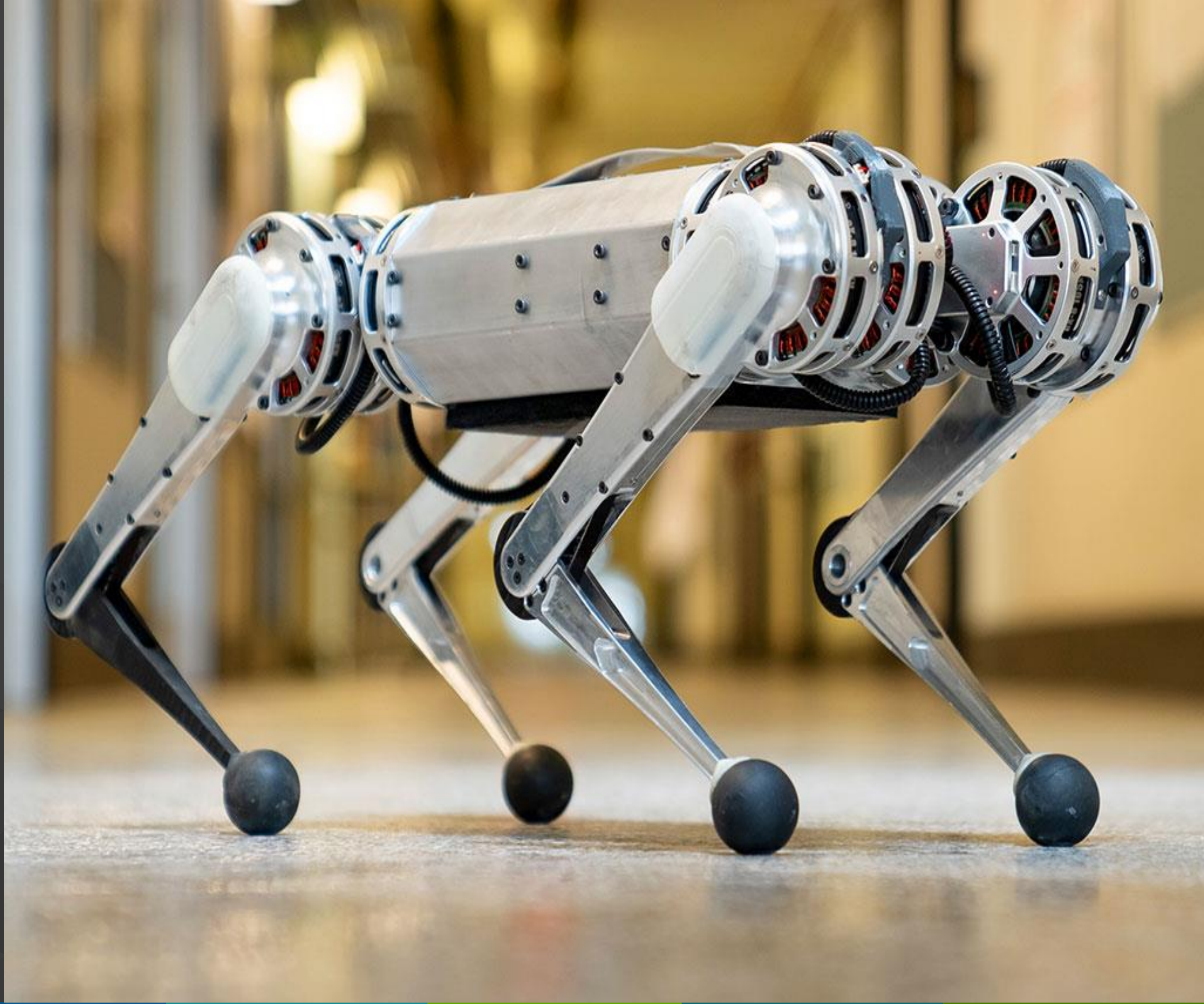
Welcome

To Your Prospect Skills

On Robotics :

Mechanics and Control



01 About ARAS

ARAS Research group originated in 1997 and is proud of its 22+ years of brilliant background, and its contributions to the advancement of academic education and research in the field of Dynamical System Analysis and Control in the robotics application. ***ARAS*** are well represented by the industrial engineers, researchers, and scientific figures graduated from this group, and numerous industrial and R&D projects being conducted in this group. The main asset of our research group is its human resources devoted all their time and effort to the advancement of science and technology. One of our main objectives is to use these potentials to extend our educational and industrial collaborations at both national and international levels. In order to accomplish that, our mission is to enhance the breadth and enrich the quality of our education and research in a dynamic environment.

Contents

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Introduction

- Preliminaries

✓ Angular Acceleration of a Rigid Body

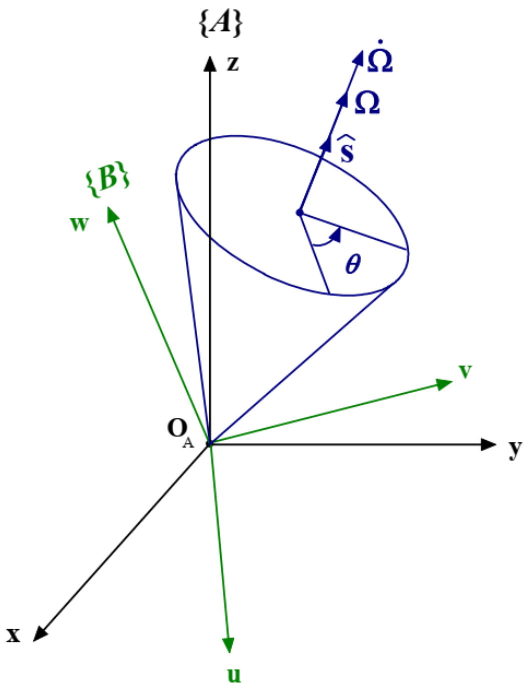
- Attribute of the whole rigid body

$$\dot{\boldsymbol{\Omega}} = \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t} = \ddot{\theta}\hat{\mathbf{s}} + \dot{\theta}\dot{\hat{\mathbf{s}}}$$
$$= \ddot{\theta}\hat{\mathbf{s}} + \dot{\theta}\left(\boldsymbol{\Omega} \times \hat{\boldsymbol{s}}\right)$$
$$= \ddot{\theta}\hat{\mathbf{s}}.$$

- Importance of screw representation

$$\boldsymbol{\Omega} \times \hat{\boldsymbol{s}} = \mathbf{0}.$$

Angular acceleration is also along $\hat{\boldsymbol{s}}$.



Preliminaries

✓ Linear Acceleration of a Point

- Start with the velocity vector of point ***P***

$$ {}^{A}\boldsymbol{v}_p = {}^{A}\boldsymbol{v}_{O_B} + {}^{A}\boldsymbol{R}_B\, {}^{B}\boldsymbol{v}_p + {}^{A}\boldsymbol{\Omega}^{\times}\, {}^{A}\boldsymbol{R}_B\, {}^{B}\boldsymbol{P}. $$

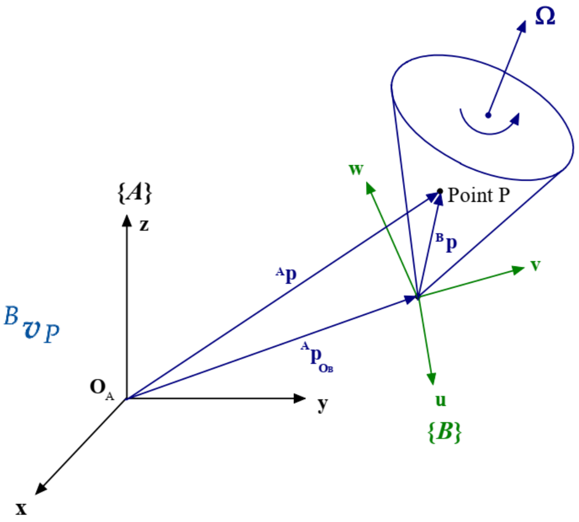
Differentiate with respect to time and manipulate

$$ {}^{A}\boldsymbol{a}_p = {}^{A}\boldsymbol{a}_{O_B} + {}^{A}\boldsymbol{R}_B\, {}^{B}\boldsymbol{a}_p + {}^{A}\dot{\boldsymbol{\Omega}}^{\times}\, {}^{A}\boldsymbol{R}_B\, {}^{B}\boldsymbol{P} + {}^{A}\boldsymbol{\Omega}^{\times}\left({}^{A}\boldsymbol{\Omega}^{\times}\, {}^{A}\boldsymbol{R}_B\, {}^{B}\boldsymbol{P}\right) + 2\, {}^{A}\boldsymbol{\Omega}^{\times}\, {}^{A}\boldsymbol{R}_B\, {}^{B}\boldsymbol{v}_P $$

Where the last two terms are centrifugal and Coriolis acceleration terms.

If ***P*** is embedded in the rigid body, the relative velocity and acceleration are equal to zero. Then

$$ {}^{A}\boldsymbol{a}_p = {}^{A}\boldsymbol{a}_{O_B} + {}^{A}\dot{\boldsymbol{\Omega}}^{\times}\, {}^{A}\boldsymbol{R}_B\, {}^{B}\boldsymbol{P} + {}^{A}\boldsymbol{\Omega}^{\times}\left({}^{A}\boldsymbol{\Omega}^{\times}\, {}^{A}\boldsymbol{R}_B\, {}^{B}\boldsymbol{P}\right). $$



Preliminaries

- Mass Properties:
  - Consider body consists of differential material bodies with density ρ and volume dV then $dm = \rho dV$ is the differential mass while the mass of the body is defined by

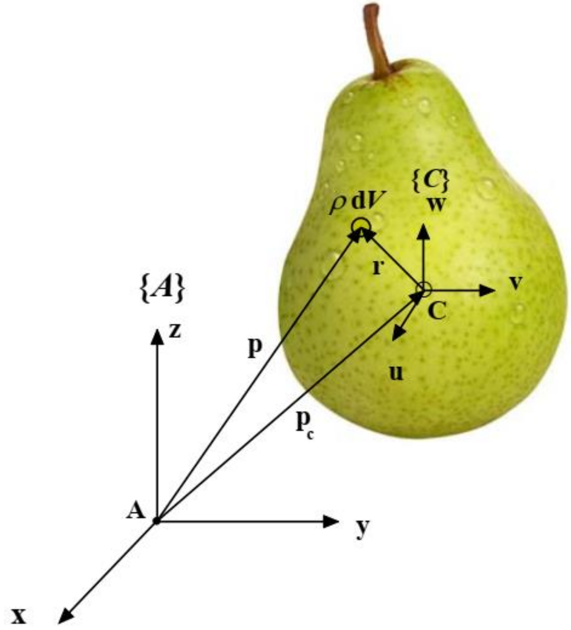
$$m = \int_V \rho \mathrm{d}V.$$

  - The center of mass $\boldsymbol{p_c}$ is then defined by

$$\boldsymbol{p}_c = \frac{1}{m}\int_V \boldsymbol{p}\,\rho \mathrm{d}V,$$

In which, $\boldsymbol{p}$ denote the position of dm

The body frame attached to the center of mass is denoted by $\{C\}$.



Preliminaries

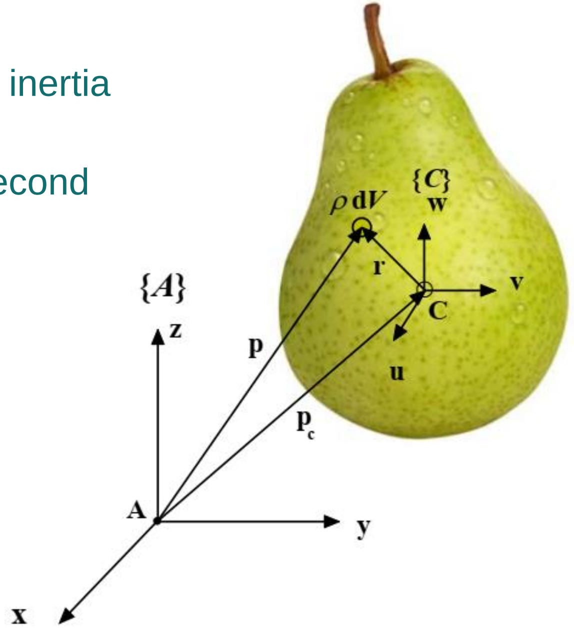
- Mass Properties:
  - ✓ Moments of Inertia
    - As opposed to the mass, moment of inertia introduces inertia to angular acceleration.
    - Moment of inertia is a tensor (matrix) defined by the second moment of mass w.r.t a frame of reference

$$ {}^{A}I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}, $$

$$ I_{xx} = \int_V (y^2 + z^2)\rho \mathrm{d}V, \quad I_{xy} = I_{yx} = -\int_V xy\, \rho \mathrm{d}V, $$

$$ I_{yy} = \int_V (x^2 + z^2)\rho \mathrm{d}V, \quad I_{yz} = I_{zy} = -\int_V yz\, \rho \mathrm{d}V, $$

$$ I_{zz} = \int_V (x^2 + y^2)\rho \mathrm{d}V, \quad I_{xz} = I_{zx} = -\int_V xz\, \rho \mathrm{d}V. $$



Preliminaries

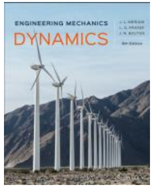
- Mass Properties:
  - ✓ Moments of Inertia
    - Principal Axes:

      A specific direction in space where the matrix is diagonal

$$ {}^{A}I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}. $$

      Principal Axes are the eigenvectors of the inertia matrix

      Physically, they are the axes of symmetry of the rigid body.



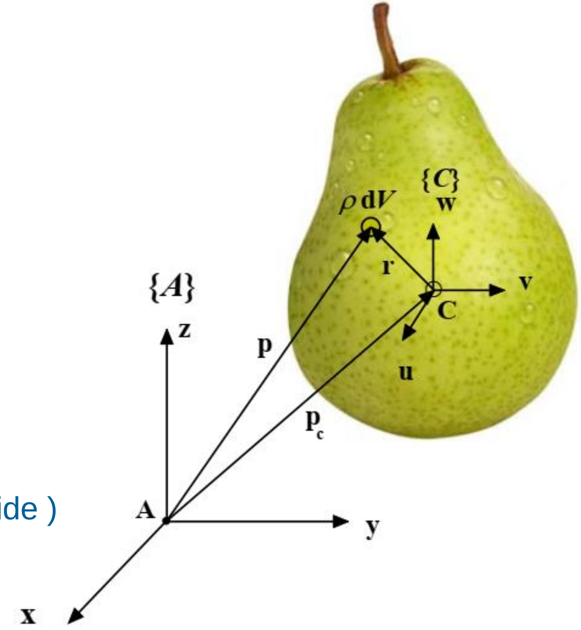
    - We never calculate the Moment of inertia by hand

      Use tables:

      Appendix D: J. L. Meriam et. Al. : Dynamics book (see next slide)

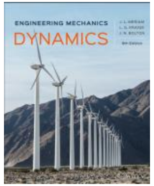
      Or Cad Softwares to calculate them.

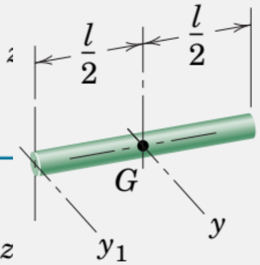
      Autocad, Solidworks, etc



Preliminaries

- Mass Properties:
  - ✓ Moments of Inertia Table:



Body		Mass Center	Mass Moments of Inertia
			
	Uniform Slender Rod	—	$I_{yy} = \frac{1}{12} ml^2$ $I_{y_1 y_1} = \frac{1}{3} ml^2$
y x		8	$\bar{I}_{yy} = \bar{I}_{zz} = \frac{83}{320} mr^2$

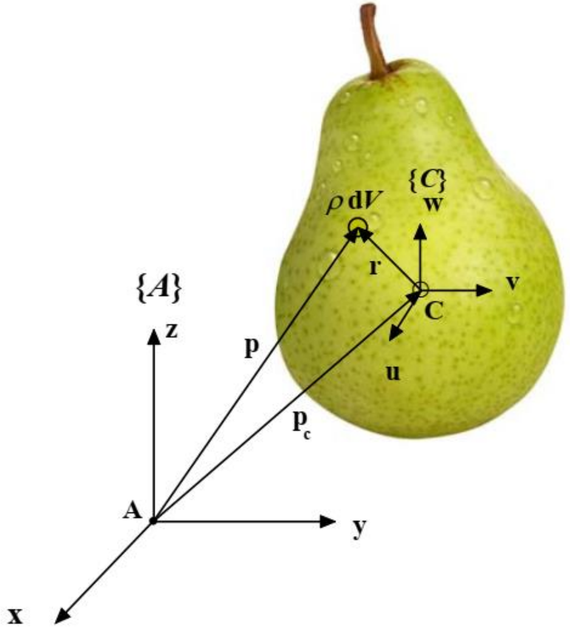
Preliminaries

- Mass Properties:
  - ✓ Inertia Matrix Transformation
    - Parallel Axis Theorem

$$ {}^{A}I = {}^{C}I + m\left(p_c^T p_c I_{3\times3} - p_c p_c^T\right), $$

Where p_c denotes the position vector of C w.r.t $\{A\}$.

$$ {}^{A}I_{xx} = {}^{C}I_{xx} + m\,(y_c^2 + z_c^2), \quad {}^{A}I_{xy} = {}^{C}I_{xy} + m\;x_c y_c, $$
$$ {}^{A}I_{yy} = {}^{C}I_{yy} + m\,(x_c^2 + z_c^2), \quad {}^{A}I_{yz} = {}^{C}I_{yz} + m\;y_c z_c, $$
$$ {}^{A}I_{zz} = {}^{C}I_{zz} + m\,(x_c^2 + y_c^2), \quad {}^{A}I_{xz} = {}^{C}I_{xz} + m\;x_c z_c. $$



    - Pure Rotation:

$$ {}^{A}I = {}^{A}\boldsymbol{R}_C\,{}^{C}I\,{}^{A}\boldsymbol{R}_C^T. $$

Preliminaries

- Momentum and Energy
  - Linear Momentum

$$^{A}\boldsymbol{G} = \int_V \frac{d\boldsymbol{p}}{dt}\rho dV \quad \rightarrow \quad {}^{A}\boldsymbol{G} = m \cdot {}^{A}\boldsymbol{v}_c$$

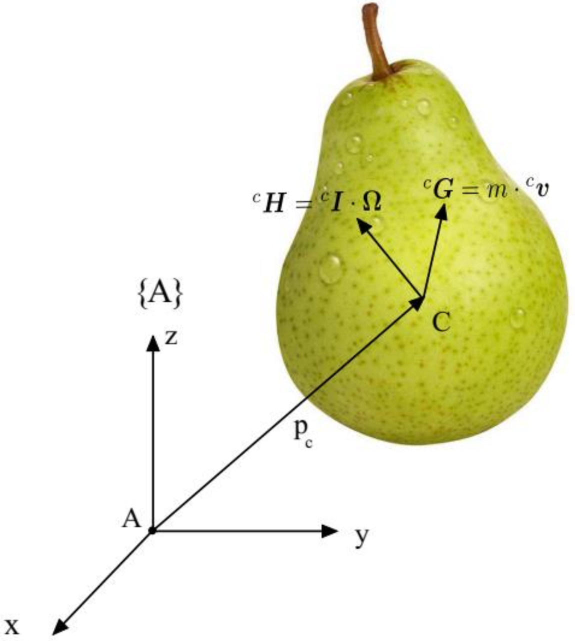
  - Angular Momentum

$$^{C}\boldsymbol{H} = \int_V \boldsymbol{r} \times (\boldsymbol{\Omega} \times \boldsymbol{r})\,\rho\, dV = {}^{C}\boldsymbol{I} \cdot \boldsymbol{\Omega}$$

$$^{A}\boldsymbol{H} = \int_V \left(\boldsymbol{p} \times \frac{d\boldsymbol{p}}{dt}\right) \rho dV, \quad \rightarrow \quad {}^{A}\boldsymbol{H} = \boldsymbol{p}_c \times \boldsymbol{G}_c + {}^{C}\boldsymbol{H}$$

  - Kinetic Energy

$$\boldsymbol{K} = \frac{1}{2}\int_V \boldsymbol{v} \cdot \boldsymbol{v} \rho\, dV \quad \rightarrow \quad \boldsymbol{k} = \frac{1}{2}\boldsymbol{v}_c \cdot \boldsymbol{G}_c + \frac{1}{2}\boldsymbol{\Omega} \cdot {}^{C}\boldsymbol{H},$$



Contents

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Lagrange Formulation

- Motivating Example
  - ✓ Consider a sprung mass pendulum as in figure
    - A pendulum with mass M, Length L, and moment of Inertia ${}^{C}I_{zz} = I$.
    - A sprung mass with mass m, and spring stiffness K
    - A motor torque τ to control the motion with no friction
    - Generalized Coordinate

      The system consists of two masses M and m → two generalized coordinate:

$$\boldsymbol{q} = \begin{bmatrix} \phi \\ r \end{bmatrix} \quad \begin{matrix} \text{Rotation} \\ \text{Translation} \end{matrix}$$

    - Generalized Forces

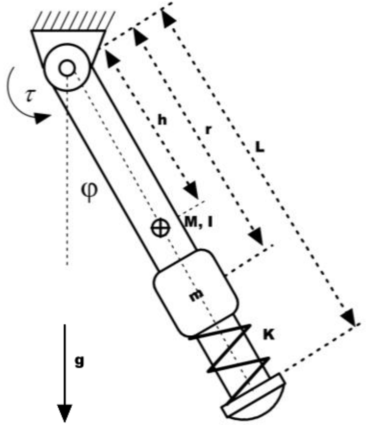
      Corresponding to each generalized coordinate, consider the external force/torques:

$$\boldsymbol{Q} = \begin{bmatrix} \tau \\ 0 \end{bmatrix} \quad \begin{matrix} \text{Torque} \\ \text{Force} \end{matrix}$$

    - Lagrangian $\mathcal{L} = K - P$

      In which K is the total kinetic energy of all rigid bodies, $K = \frac{1}{2} m \boldsymbol{v}_{\boldsymbol{c}}^{\mathbf{2}} + \frac{1}{2}\, {}^{C}I \boldsymbol{\omega}^{\mathbf{2}}$

      while, P is the total potential energy w.r.t a reference level. $P = mgh + \frac{1}{2} k x^2$



Lagrange Formulation

- Motivating Example
  - ✓ Governing Dynamic Equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j, \qquad j = 1, 2, \ldots, m$$

- Where, q_i denotes the generalized coordinate, Q_i denotes the generalized force, and m denotes the number of independent generalized coordinate

  - ✓ The key point is
    - To derive the angular and center of mass velocities correctly.

$$v_{c_1}^2 = \left(h\dot{\phi}\right)^2 \quad \omega_1 = \dot{\phi} \text{ and } v_{c_2}^2 = \dot{r}^2 + \left(r\dot{\phi}\right)^2 \quad \omega_2 = \dot{\phi}$$

Hence the total kinetic energy is

$$K = \frac{1}{2}M\left(h\dot{\phi}\right)^2 + \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m\left[\dot{r}^2 + \left(r\dot{\phi}\right)^2\right]$$

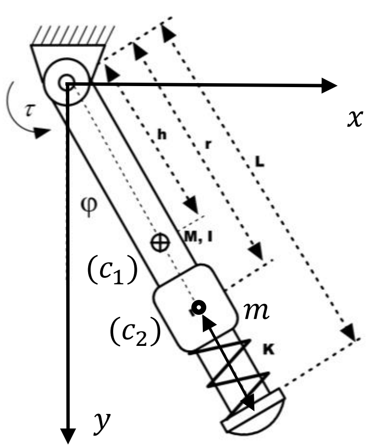
Notice that the mass m is considered as a point mass with no moment of inertia.

The total potential energy is (r_o denotes the position where the spring is at equilibrium):

$$P = -Mgh\cos\phi - mgr\cos\phi + \frac{1}{2}k(r - r_o)^r$$

Hence:

$$\mathcal{L} = \frac{1}{2}(Mh^2 + mr^2 + I)\dot{\phi}^2 + \frac{1}{2}m\dot{r}^2 + (Mh + mr)g\cos\phi + \frac{1}{2}k(r - r_0)^2$$



Lagrange Formulation

- Motivating Example
  - ✓ Governing Dynamic Equations:
    - For $q_1 = \phi$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \left(\frac{\partial \mathcal{L}}{\partial \phi}\right) = Q_1$$

$$\mathcal{L} = \frac{1}{2}(Mh^2 + mr^2 + I)\dot{\phi}^2 + \frac{1}{2}m\dot{r}^2 + (Mh + mr)g\cos\phi + \frac{1}{2}k(r - r_0)^2$$

    - This yields to:

$$\frac{d}{dt}\left((Mh^2 + mr^2 + I)\dot{\phi}\right) + (Mh + mr)g\sin\phi = \tau$$

$$(Mh^2 + mr^2 + I)\ddot{\phi} + 2mr\dot{r}\dot{\phi} + (Mh + mr)g\sin\phi = \tau$$

    - For $q_2 = r$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) - \left(\frac{\partial \mathcal{L}}{\partial r}\right) = Q_2$$

    - This yields to:

$$m\ddot{r} - mr\dot{\phi} - mrgcos\,\phi k(r - r_0) = 0$$

Lagrange Formulation

- Generalized Coordinates and Forces
  - ✓ Serial robot with primary joints
    - Each joint introduces a generalized coordinate:

$$\boldsymbol{q} = [q_1, q_2, \dots, q_n]^T \text{ in which } q_i = \begin{cases} \theta_i & \text{for a revolute joint} \\ d_i & \text{for a prismatic joint} \end{cases}$$

    - Each joint actuator introduces a generalized force:

$$\boldsymbol{Q} = [Q_1, Q_2, \dots, Q_n]^T \text{ in which } Q_i = \begin{cases} \tau_i & \text{for a revolute joint} \\ f_i & \text{for a prismatic joint} \end{cases}$$

If the robot is applying a wrench $\boldsymbol{\mathcal{F}_e}$ to the environment, Then

$$\boldsymbol{Q} = \boldsymbol{\tau} + \boldsymbol{J}^T \boldsymbol{\mathcal{F}_e}$$

If the joints have friction:

$$Q_i = \begin{cases} \tau_i - \tau_f & \text{for a revolute joint} \\ f_i - f_f & \text{for a prismatic joint} \end{cases}$$

In which, τ_f / f_f denotes the joint friction torque/force.

    - They are called generalized: To include both rotational/translational motion and torque/force actuation.

Lagrange Formulation

- Kinetic Energy
  - ✓ Serial robot with primary joints
    - For joint i: $K_i = \frac{1}{2} m_i \boldsymbol{v}_{c_i}^T \boldsymbol{v}_{c_i} + \frac{1}{2} \boldsymbol{\omega}_i^T \, {}^C I_i \boldsymbol{\omega}_i$
    - Total Kinetic Energy: $K = \sum_{i=1}^{n} K_i$
    - The velocity vectors and the moment of inertia could be expressed in any frame.

      Inertia about center of mass is simply found in the moving frame ${}^i I_i$ (and it is constant)

      To express it in the base frame: ${}^C\boldsymbol{I}_i = {}^0\boldsymbol{R}_i \; {}^i\boldsymbol{I}_i \; {}^0\boldsymbol{R}_i^T$.
    - The velocity $\boldsymbol{v}_{c_i}$ can be found by conventional or screw-based Jacobian mapping

      For conventional Jacobian: $\dot{\boldsymbol{\chi}}_{c_i} = \boldsymbol{J}_i \dot{\boldsymbol{q}}$

      Where $\dot{\boldsymbol{\chi}}_{c_i} = \begin{bmatrix} \boldsymbol{v}_{c_i} \\ \omega_i \end{bmatrix}$ and $\boldsymbol{J}_i = \begin{bmatrix} \boldsymbol{J}_{v_i} \\ \boldsymbol{J}_{\omega_i} \end{bmatrix}$.

      In which J_i denotes the link Jacobian

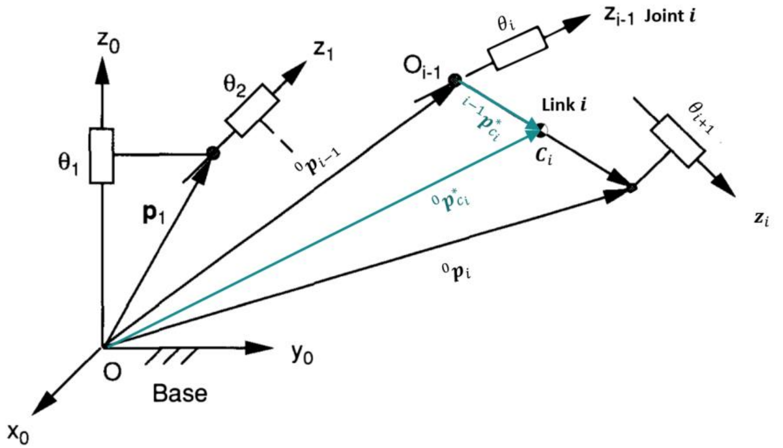
      and J_{v_i} and J_{ω_i} denote the translational/rotational Jacobian submatrices

Lagrange Formulation

- Kinetic Energy
  - ✓ To derive the link Jacobians $\boldsymbol{J_i} = \begin{bmatrix} \boldsymbol{J_{v_i}} \\ \boldsymbol{J_{\omega_i}} \end{bmatrix}$
    - Use the general derivation method for Jacobian column vectors $J_{v_i}^{\mathrm{j}}$ and $J_{\omega_i}^{\mathrm{j}}$ for $\boldsymbol{j} \le \boldsymbol{i}$.

$$J_{v_i}^{\mathrm{j}} = \begin{cases} \boldsymbol{z}_{j-1} \times {}^{j-1}\boldsymbol{p}_{c_i}^{*} & \text{revolute} \\ \boldsymbol{z}_{j-1} & \text{prismatic} \end{cases} \quad \text{and} \quad J_{\omega_i}^{\mathrm{j}} = \begin{cases} \boldsymbol{z}_{j-1} & \text{revolute} \\ \boldsymbol{0} & \text{prismatic} \end{cases}$$

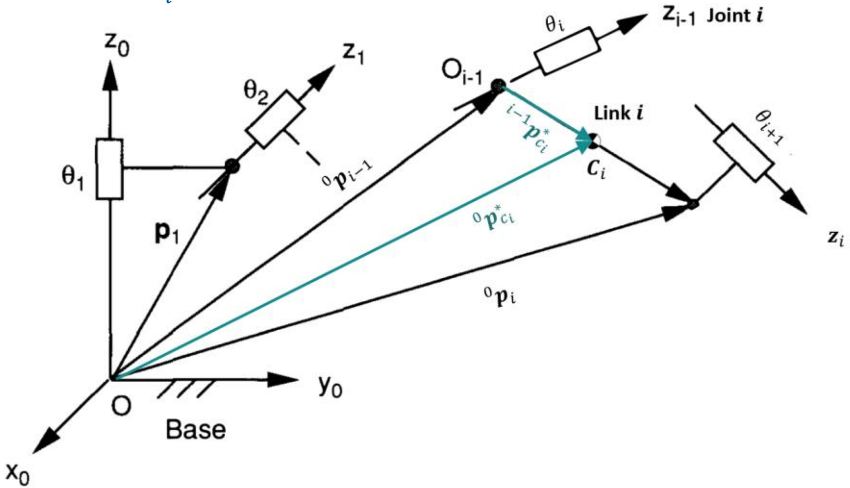
Where ${}^{j-1}\boldsymbol{p}_{c_i}^{*}$ is the position vector from the origin of $j-1$ link frame to the center of mass of link i and expressed in the base frame.



Lagrange Formulation

- Kinetic Energy

  The procedure is like what on regular Jacobians, but the ${}^0\boldsymbol{p}_n^*$ is replaced with the position vector of the center of mass ${}^0\boldsymbol{p}_{c_i}^*$. (Chapter 04 – Slide 21)



- Note that the velocity v_{c_i} depends only to the joint velocities 1 to i, and therefore, $J_{v_i}^j = J_{\omega_i}^j = 0$ for $\boldsymbol{j} > \boldsymbol{i}$.

$$J_{v_i} = \left[J_{v_i}^1, J_{v_i}^2, \dots, J_{v_i}^j, \mathbf{0}, \mathbf{0}, \dots, \mathbf{0}\right] \quad , \quad J_{\omega_i} = \left[J_{\omega_i}^1, J_{\omega_i}^2, \dots, J_{\omega_i}^i, \mathbf{0}, \mathbf{0}, \dots, \mathbf{0}\right]$$

Lagrange Formulation

- Kinetic Energy
  - Therefore, $K = \frac{1}{2}\sum_{i=1}^{n}(m_i \boldsymbol{v}_{c_i}^T \boldsymbol{v}_{c_i} + \boldsymbol{\omega}_i^T \, {}^C I_i \boldsymbol{\omega}_i)$
  - Use Jacobians: $= \frac{1}{2}\dot{\boldsymbol{q}}^T \left\{\sum_{i=1}^{n}(m_i \boldsymbol{J}_{\boldsymbol{v}_i}^{\boldsymbol{T}} \boldsymbol{J}_{\boldsymbol{v}_i} + \boldsymbol{J}_{\boldsymbol{\omega}_i}^{\boldsymbol{T}} \, {}^C I_i \, \boldsymbol{J}_{\boldsymbol{\omega}_i})\right\} \dot{\boldsymbol{q}}$
  - Define Mass Matrix: $$\boldsymbol{M}(\boldsymbol{q}) = \sum_{i=1}^{n}(m_i \boldsymbol{J}_{\boldsymbol{v}_i}^{\boldsymbol{T}} \boldsymbol{J}_{\boldsymbol{v}_i} + \boldsymbol{J}_{\boldsymbol{\omega}_i}^{\boldsymbol{T}} \, {}^C I_i \, \boldsymbol{J}_{\boldsymbol{\omega}_i})$$
  - By this means: $K = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$

    The Mass Matrix is a symmetric, and positive definite matrix.

    The kinetic energy is also always positive unless the system is at rest.
- Potential Energy
  - The work required to displace link i to position $\boldsymbol{p}_{c_i}$ is given by $-m_i \boldsymbol{g}^T \boldsymbol{p}_{c_i}$, in which $\boldsymbol{g}$ is the vector of gravity acceleration, Therefore,
    $$P = -\sum_{i=1}^{n} m_i \boldsymbol{g}^T \, {}^{\boldsymbol{0}}\boldsymbol{p}_{c_i}$$
  - If there is any spring (flexibility) in joints, then
    $$P = -\sum_{i=1}^{n} m_i \boldsymbol{g}^T \, {}^{\boldsymbol{0}}\boldsymbol{p}_{c_i} + \frac{1}{2}\sum_{i=1}^{n} k_i (\Delta x)^2$$

    In which k_i denote the spring stiffness while Δx denote its deflection

Lagrange Formulation

- General Formulation
  - ✓ Consider nDoF serial manipulator with primary joints
    - Denote $\boldsymbol{q} = [q_1, q_2, \dots, q_n]^{\boldsymbol{T}}$ as the generalized coordinates and
    - Denote $\boldsymbol{Q} = [Q_1, Q_2, \dots, Q_n]^{\boldsymbol{T}}$ as the generalized Forces.
    - Form Lagrangian by $\mathcal{L} = K - P$
    - Derive the governing equation of motion in vector form by:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} = \boldsymbol{Q}$$

in which, $\mathcal{L} = K(\boldsymbol{q}, \dot{\boldsymbol{q}}) - P(\boldsymbol{q})$

hence, $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right) = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\boldsymbol{q}}}\right) = \frac{d}{dt}\left(M(q)\dot{\boldsymbol{q}}\right) = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}}$

Furthermore, define the gravity vector as $\mathbf{g}(\boldsymbol{q}) = \frac{\partial P}{\partial \boldsymbol{q}} = -\sum_{i=1}^{n} \frac{\partial P}{\partial \boldsymbol{q}}\left(m_i \boldsymbol{g}^T\, {}^{\mathbf{0}}\boldsymbol{p}_{c_i}\right) = -\sum_{i=1}^{n} m_i \boldsymbol{J}_{\boldsymbol{v}_i}^{\boldsymbol{T}} \boldsymbol{g}$.

In which, $\boldsymbol{J}_{v_i}^{\boldsymbol{T}}$ denotes the linear velocity Jacobian of each center of motion. Hence the dynamics is written as:

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \left(\dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{\partial K}{\partial \boldsymbol{q}}\right) + \mathbf{g}(\boldsymbol{q}) = \boldsymbol{Q}$$

Lagrange Formulation

- General Formulation
  - ✓ This can be written in general form of

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \mathbf{v}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \mathbf{g}(\boldsymbol{q}) = \boldsymbol{Q}$$

  - In which $\mathbf{v}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is called the Coriolis and centrifugal vector

$$\mathbf{v}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{\partial K}{\partial \boldsymbol{q}} = \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{q}}\left(\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}\right)$$

This vector may be written in a Christoffel matrix form $\mathbf{v}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \mathbf{g}(\boldsymbol{q}) = \boldsymbol{Q}$$

The derivation and properties of dynamic matrices are elaborated next.

Lagrange Formulation

- Examples:
  - ✓ Example 1: Planar Elbow Manipulator
    - Consider

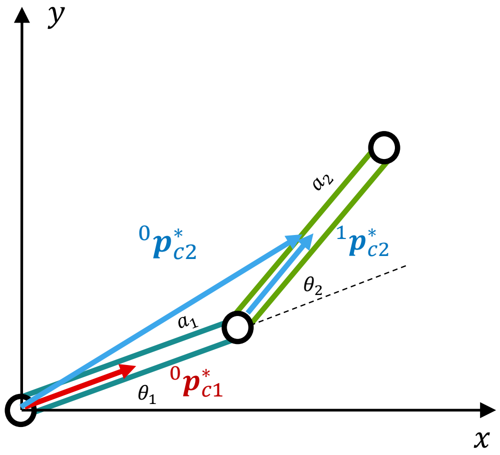
      slender bar links with mass m_i @ their half length,

      viscous friction $\tau_{f_i} = b_i \dot{q}_i$ at the joints

      No joint flexibility, and no applied wrench.
    - Denote $\boldsymbol{q} = [\theta_1, \theta_2]^T$ and $\boldsymbol{Q} = [\tau_1 - b_1\dot{q}_1, \tau_2 - b_2\dot{q}_2]^T$
    - The inertia in z direction is found by

      ${}^iI_{zz_i} = \frac{1}{12} m_i a_i^2$ which is the same in base frame: ${}^CI_{zz_i} = \frac{1}{12} m_i a_i^2$
    - The position vector of the center of mass in base frame:

$$
{}^0\boldsymbol{p}_{c1}^* = \frac{1}{2} a_1 \begin{bmatrix} c_1 \\ s_1 \\ 0 \end{bmatrix}, \ {}^1\boldsymbol{p}_{c2}^* = \frac{1}{2} a_2 \begin{bmatrix} c_{12} \\ s_{12} \\ 0 \end{bmatrix} \rightarrow {}^0\boldsymbol{p}_{c2}^* = \begin{bmatrix} a_1 c_1 + \frac{1}{2} a_2 c_{12} \\ a_1 s_1 + \frac{1}{2} a_2 s_{12} \\ 0 \end{bmatrix}
$$



Matlab Code: Lagrange_2R.m

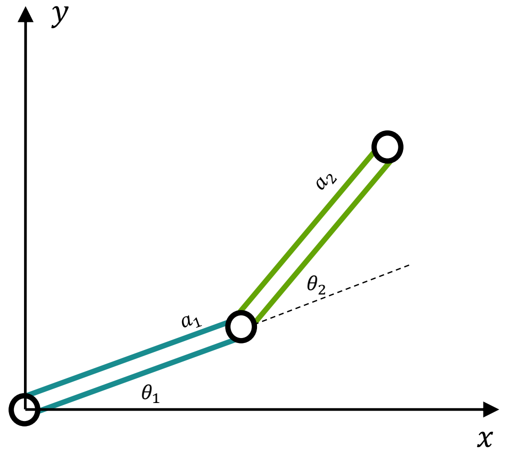
Lagrange Formulation

✓ Example 1: Planar Elbow Manipulator

- The link Jacobians are derived easily as:

$$J_{v_1} = \frac{1}{2} a_1 \begin{bmatrix} -s_1 & 0 \\ c_1 & 0 \\ 0 & 0 \end{bmatrix}, \; J_{\omega_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$J_{v_2} = \begin{bmatrix} -a_1 s_1 - \frac{1}{2} a_2 s_{12} & -\frac{1}{2} a_2 s_{12} \\ a_1 c_1 + \frac{1}{2} a_2 c_{12} & \frac{1}{2} a_2 c_{12} \\ 0 & 0 \end{bmatrix}, \; J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$



Matlab Code: Lagrange_2R.m

- The robot Mass Matrix is derived as:

$$M(q) = \sum_{i=1}^{2} \left(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T \, {}^C I_i \, J_{\omega_i} \right) = \begin{bmatrix} \frac{1}{3} m_1 a_1^2 + m_2 \left(a_1^2 + a_1 a_2 c_2 + \frac{1}{3} a_2^2 \right) & m_2 \left(\frac{1}{2} a_1 a_2 c_2 + \frac{1}{3} a_2^2 \right) \\ m_2 \left(\frac{1}{2} a_1 a_2 c_2 + \frac{1}{3} a_2^2 \right) & \frac{1}{3} m_2 a_2^2 \end{bmatrix}$$

Lagrange Formulation

✓ Example 1: Planar Elbow Manipulator

- The Gravity vector is derived as:

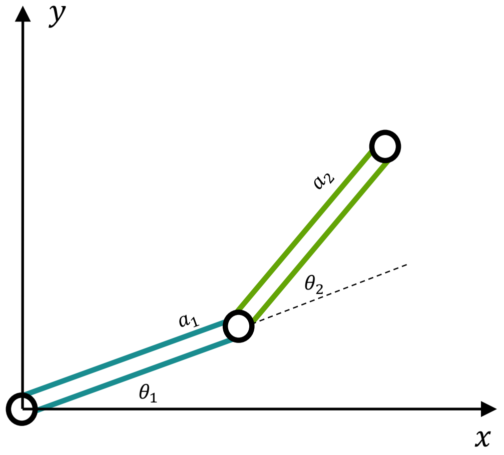
$$\mathbf{g}(\boldsymbol{q}) = -\sum_{i=1}^{2} m_i J_{v_i}^T \boldsymbol{g} = \begin{bmatrix} \left(\frac{1}{2}m_1 + m_2\right) g a_1 c_1 + \frac{1}{2} m_2 g a_2 c_{12} \\ \frac{1}{2} m_2 g a_2 c_{12} \end{bmatrix}$$

- The Coriolis and centrifugal vector is derived as:

$$\mathbf{v}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{\partial K}{\partial \boldsymbol{q}} = \begin{bmatrix} -m_2 a_1 a_2 s_2 \left(\dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2}\dot{\theta}_2^2\right) \\ \frac{1}{2} m_2 a_1 a_2 s_2 \dot{\theta}_1^2 \end{bmatrix}$$

- The Lagrange equations is formulates by:

$$\mathbf{Q} = \begin{bmatrix} \tau_1 - b_1 \dot{q}_2 \\ \tau_2 - b_2 \dot{q}_2 \end{bmatrix} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \mathbf{v}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \mathbf{g}(\boldsymbol{q}).$$



Matlab Code: Lagrange_2R.m

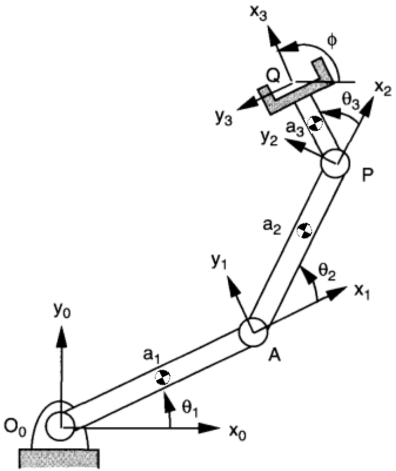
Lagrange Formulation

✓ Example 2: Planar RRR Manipulator

- Like example 1 consider:

  slender bar links with mass m_i @ their half length with no friction, no joint flexibility, and no applied wrench.

- The position vector of the center of mass in base frame:



$$ {}^0\boldsymbol{p}_{c1}^* = \frac{1}{2}a_1\begin{bmatrix} c_1 \\ s_1 \\ 0 \end{bmatrix}, \; {}^1\boldsymbol{p}_{c2}^* = \frac{1}{2}a_2\begin{bmatrix} c_{12} \\ s_{12} \\ 0 \end{bmatrix} \to {}^0\boldsymbol{p}_{c2}^* = \begin{bmatrix} a_1c_1 + \frac{1}{2}a_2c_{12} \\ a_1s_1 + \frac{1}{2}a_2s_{12} \\ 0 \end{bmatrix} $$

$$ {}^2\boldsymbol{p}_{c3}^* = \frac{1}{2}a_3\begin{bmatrix} c_{123} \\ s_{123} \\ 0 \end{bmatrix} \to {}^1\boldsymbol{p}_{c3}^* = \begin{bmatrix} a_2c_{12} + \frac{1}{2}a_3c_{123} \\ a_2s_{12} + \frac{1}{2}a_3s_{123} \\ 0 \end{bmatrix} \text{ and } {}^0\boldsymbol{p}_{c3}^* = \begin{bmatrix} a_1c_1 + a_2c_{12} + \frac{1}{2}a_3c_{123} \\ a_1s_1 + a_2s_{12} + \frac{1}{2}a_3s_{123} \\ 0 \end{bmatrix} $$

Matlab Code: Lagrange_3R.m

Lagrange Formulation

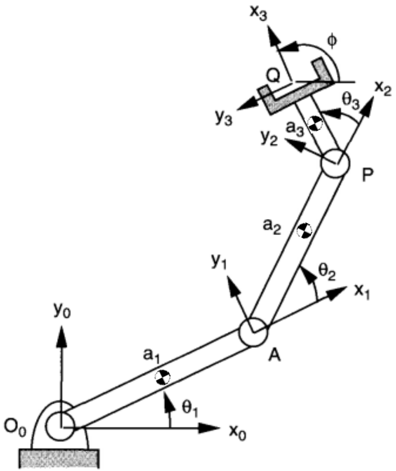
✓ Example 2: Planar RRR Manipulator

- The link Jacobians are derived easily as:

$$J_{v_1} = \frac{1}{2}a_1\begin{bmatrix} -s_1 & 0 & 0 \\ c_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, J_{\omega_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$J_{v_2} = \begin{bmatrix} -a_1s_1 - \frac{1}{2}a_2s_{12} & -\frac{1}{2}a_2s_{12} & 0 \\ a_1c_1 + \frac{1}{2}a_2c_{12} & \frac{1}{2}a_2c_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}, J_{\omega_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$J_{v_3} = \begin{bmatrix} -a_1s_1 - a_2s_{12} - \frac{1}{2}a_3s_{123} & -a_2s_{12} - \frac{1}{2}a_3s_{123} & -\frac{1}{2}a_3s_{123} \\ a_1c_1 + a_2c_{12} + \frac{1}{2}a_3cs_{123} & a_2c_{12} + \frac{1}{2}a_3c_{123} & \frac{1}{2}a_3c_{123} \\ 0 & 0 & 0 \end{bmatrix}, J_{\omega_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$



Matlab Code: Lagrange_3R.m

Lagrange Formulation

✓ Example 2: Planar RRR Manipulator

- The robot Mass Matrix is derived as:

$$\boldsymbol{M}_{3\times3}(\boldsymbol{q}) = \sum_{i=1}^{2}\left(m_i \boldsymbol{J}_{v_i}^T \boldsymbol{J}_{v_i} + \boldsymbol{J}_{\omega_i}^T \, {}^{C}I_i \, \boldsymbol{J}_{\omega_i}\right)$$

In which,

$$M_{11} = \left(\frac{1}{3}m_1 + m_2 + m_3\right)a_1^2 + \left(\frac{1}{3}m_2 + m_3\right)a_2^2 + \frac{1}{3}m_3a_3^2 + (m_2 + 2m_3)a_1a_2c_2 + a_3m_3(a_1c_{23} + a_2c_3)$$

$$M_{22} = \left(\frac{1}{3}m_2 + m_3\right)a_2^2 + \frac{1}{3}m_3a_3^2 + m_3a_2a_3c_3, \quad M_{33} = \frac{1}{3}m_3a_3^2$$

$$M_{12} = M_{21} = \left(\frac{1}{3}m_2 + m_3\right)a_2^2 + m_3a_3\left(\frac{1}{3}a_3 + \frac{1}{2}a_1c_{23}\right) + \frac{1}{2}m_2a_1a_2c_2 + m_3a_2(a_1c_2 + a_3c_3)$$

$$M_{23} = M_{32} = m_3a_3\left(\frac{1}{3}a_3 + \frac{1}{2}a_2c_3\right), \quad M_{13} = M_{31} = m_3a_3\left(\frac{1}{3}a_3 + \frac{1}{2}a_1c_{23} + \frac{1}{2}a_2c_3\right)$$

- The Gravity vector is derived as:

$$\mathbf{g}(\boldsymbol{q}) = \begin{bmatrix} \frac{1}{2}m_1a_1g\,c_1 + \frac{1}{2}m_2g\,(a_1c_1 + a_2c_{12}) + m_3g\left(a_2c_{12} + \frac{1}{2}a_3c_{123}\right) \\ \frac{1}{2}m_2a_2g\,c_{12} + m_3g\left(a_2c_{12} + \frac{1}{2}a_3c_{123}\right) \\ \frac{1}{2}m_3a_3g\,c_{123} \end{bmatrix}$$

Matlab Code: Lagrange_3R.m

Lagrange Formulation

✓ Example 2: Planar RRR Manipulator

- The Coriolis and centrifugal vector is derived as:

$$\boldsymbol{V}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\frac{\partial}{\partial q}\left(\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}\right) = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

In which,

$$V_1 = -\dot{\theta}_1\left(\dot{\theta}_2(a_1a_3m_3s_{23} + a_1a_2m_2s_2 + 2a_1a_2m_3s_2) + \dot{\theta}_3(a_1a_3m_3s_{23} + a_2a_3m_3s_3)\right)$$
$$-\dot{\theta}_2\left(\dot{\theta}_2\left(\frac{1}{2}a_1a_3m_3s_{23} + \frac{1}{2}a_1a_2m_2s_2 + a_1a_2m_3s_2\right) + \dot{\theta}_3\left(\frac{1}{2}a_1a_3m_3s_{23} + a_2a_3m_3s_3\right)\right)$$
$$-\dot{\theta}_3\left(a_3m_3\dot{\theta}_3\left(\frac{1}{2}a_1s_{23} + \frac{1}{2}a_2s_3\right) + \frac{1}{2}a_1a_3m_3s_{23}\dot{\theta}_2\right)$$
$$V_2 = \frac{1}{2}(a_1a_3m_3s_{23} + a_1a_2m_2s_2 + a_1a_2m_3s_2)\dot{\theta}_1^2 - \frac{1}{2}\left(a_2a_3m_3s_3\dot{\theta}_3^2\right) - a_2a_3m_3s_3\left(\dot{\theta}_1\dot{\theta}_3 + \dot{\theta}_2\dot{\theta}_3\right)$$
$$V_3 = \frac{1}{2}\left(m_3a_3((a_1s_{23} + a_2s_3)\dot{\theta}_1^2 + a_2s_3\dot{\theta}_2^2 + 2a_2s_3\dot{\theta}_1\dot{\theta}_2\right)$$

- The Lagrange equations is formulates by:

$$\mathbf{Q} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{bmatrix} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{V}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{G}(\boldsymbol{q}).$$

Matlab Code: Lagrange_3R.m

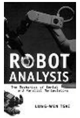
Lagrange Formulation

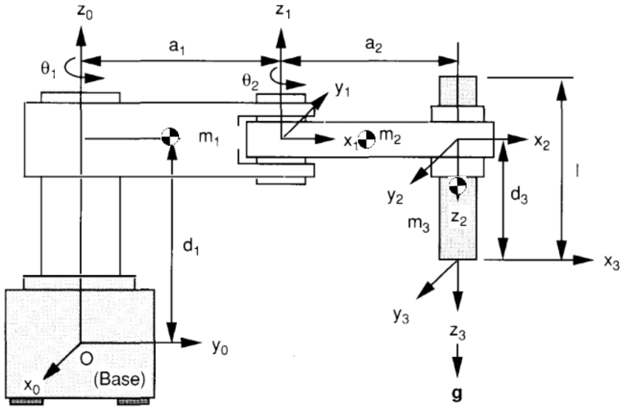
- ✓ Example 3: SCARA Arm
  - Like example 1 consider:

    slender bar links with mass m_i @ their half length with no friction, no joint flexibility, and no applied wrench.
  - The inertia matrix is found by

    $$ {}^1I_{yy_1} = {}^1I_{zz_1} = \frac{1}{12}m_1 a_1^2 \ ; \ \ {}^2I_{yy_2} = {}^2I_{zz_2} = \frac{1}{12}m_2 a_2^2 \ ,$$

    $$ {}^3I_{xx_3} = {}^3I_{yy_3} = \frac{1}{12}m_3 \ell^2 \ .$$
  - The position vector of the center of mass in base frame:





$$ {}^0\boldsymbol{p}_{c1}^* = \begin{bmatrix} \frac{1}{2}a_1 c_1 \\ \frac{1}{2}a_1 s_1 \\ d_1 \end{bmatrix}, \ {}^1\boldsymbol{p}_{c2}^* = \begin{bmatrix} \frac{1}{2}a_2 c_{12} \\ \frac{1}{2}a_2 s_{12} \\ 0 \end{bmatrix} \to {}^0\boldsymbol{p}_{c2}^* = \begin{bmatrix} a_1 c_1 + \frac{1}{2}a_2 c_{12} \\ a_1 s_1 + \frac{1}{2}a_2 s_{12} \\ d_1 \end{bmatrix}$$

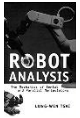
Matlab Code: Lagrange_SCARA.m

$$ {}^2\boldsymbol{p}_{c3}^* = \begin{bmatrix} 0 \\ 0 \\ d_3 - \frac{1}{2}\ell \end{bmatrix} \to {}^1\boldsymbol{p}_{c3}^* = \begin{bmatrix} a_2 c_{12} \\ a_2 s_{12} \\ -d_3 + \frac{1}{2}\ell \end{bmatrix} \to {}^0\boldsymbol{p}_{c3}^* = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ d_1 - d_3 + \frac{1}{2}\ell \end{bmatrix}$$

Lagrange Formulation

✓ Example 3: SCARA Arm

- The link Jacobians are derived easily as:



$$J_{v_1} = \frac{1}{2} a_1 \begin{bmatrix} -s_1 & 0 & 0 \\ c_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, J_{\omega_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$J_{v_2} = \begin{bmatrix} -a_1 s_1 - \frac{1}{2} a_2 s_{12} & -\frac{1}{2} a_2 s_{12} & 0 \\ a_1 c_1 + \frac{1}{2} a_2 c_{12} & \frac{1}{2} a_2 c_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix}, J_{\omega_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

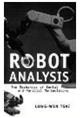
$$J_{v_3} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} & 0 \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}, J_{\omega_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Matlab Code: Lagrange_SCARA.m

Lagrange Formulation

✓ Example 3: SCARA Arm

- The robot Mass Matrix is derived as:



$$\boldsymbol{M}(\boldsymbol{q}) = \sum_{i=1}^{2}\left(m_i \boldsymbol{J}_{v_i}^{T}\boldsymbol{J}_{v_i} + \boldsymbol{J}_{\omega_i}^{T}\,{}^{C}I_i\,\boldsymbol{J}_{\omega_i}\right)$$

$$= m_1\begin{bmatrix} \frac{1}{3}a_1^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + m_2\begin{bmatrix} a_1^2 + a_1a_2c_2 + \frac{1}{3}a_2^2 & \frac{1}{2}a_1a_2c_2 + \frac{1}{3}a_2^2 & 0 \\ \frac{1}{2}a_1a_2c_2 + \frac{1}{3}a_2^2 & \frac{1}{3}a_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} +$$

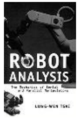
$$m_3\begin{bmatrix} a_1^2 + 2a_1a_2c_2 + a_2^2 & a_1a_2c_2 + a_2^2 & 0 \\ a_1a_2c_2 + a_2^2 & a_2^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matlab Code: Lagrange_SCARA.m

$$M = \begin{bmatrix} \frac{1}{3}m_1a_1^2 + m_2\left(a_1^2 + \frac{1}{3}a_2^2 + a_1a_2c_2\right) + m_3\left(a_1^2 + a_2^2 + \frac{1}{12}a_3^2 + 2a_1a_2c_2\right) & -m_2a_2\left(\frac{1}{3}a_2 + \frac{1}{2}a_1c_2\right) - m_3\left(a_2^2 + \frac{1}{12}a_3^2 + a_1a_2c_2\right) & 0 \\ -m_2a_2\left(\frac{1}{3}a_2 + \frac{1}{2}a_1c_2\right) - m_3\left(a_2^2 + \frac{1}{12}a_3^2 + a_1a_2c_2\right) & \frac{1}{3}m_2a_2^2 + m_3\left(a_2^2 + \frac{1}{12}a_3^2\right) & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

Lagrange Formulation

✓ Example 3: SCARA Arm

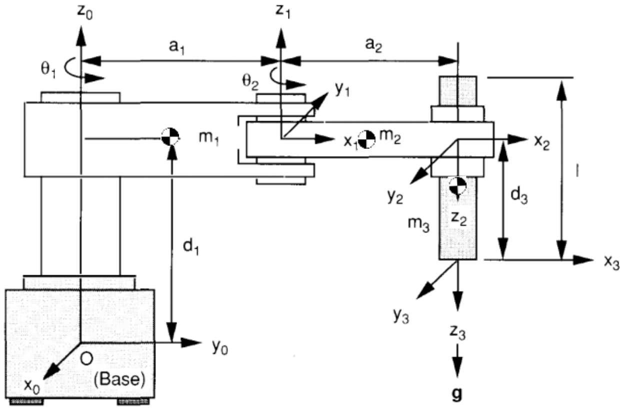


- The Gravity vector is derived as:

$$\mathbf{g}(\boldsymbol{q}) = \begin{bmatrix} 0 \\ 0 \\ -m_3 g \end{bmatrix}$$

- The Coriolis and centrifugal vector is derived as:

$$\mathbf{v}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \begin{bmatrix} -(m_2+2m_3)a_1 a_2 s_2 \left(\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}\dot{\theta}_2^2\right) \\ \left(\frac{1}{2}m_2 + m_3\right) a_1 a_2 s_2 \dot{\theta}_1^2 \\ 0 \end{bmatrix}$$



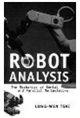
Matlab Code: Lagrange_SCARA.m

- The Lagrange equations is formulated by:

$$\mathbf{Q} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ f_3 \end{bmatrix} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \mathbf{v}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \mathbf{g}(\boldsymbol{q}).$$

Lagrange Formulation

✓ Example 3: SCARA Arm



- The Lagrange equations is formulated by:

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \mathbf{v}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \mathbf{g}(\boldsymbol{q}) = \boldsymbol{Q}$$

$$\begin{aligned}
\tau_1 = {} & \left[\left(\tfrac{1}{3}m_1 + m_2 + m_3\right)a_1^2 + (m_2 + 2m_3)a_1a_2c\theta_2 + \left(\tfrac{1}{3}m_2 + m_3\right)a_2^2\right]\ddot{\theta}_1 \\
& + \left[\left(\tfrac{1}{2}m_2 + m_3\right)a_1a_2c\theta_2 + \left(\tfrac{1}{3}m_2 + m_3\right)a_2^2\right]\ddot{\theta}_2 \\
& - (m_2 + 2m_3)a_1a_2s\theta_2\left(\dot{\theta}_1\dot{\theta}_2 + \tfrac{1}{2}\dot{\theta}_2^2\right), \\
\tau_2 = {} & \left[\left(\tfrac{1}{2}m_2 + m_3\right)a_1a_2c\theta_2 + \left(\tfrac{1}{3}m_2 + m_3\right)a_2^2\right]\ddot{\theta}_1 + \left(\tfrac{1}{3}m_2 + m_3\right)a_2^2\ddot{\theta}_2 \\
& + \left(\tfrac{1}{2}m_2 + m_3\right)a_1a_2s\theta_2\dot{\theta}_1^2, \\
f_3 = {} & m_3\ddot{d}_3 - m_3g_c.
\end{aligned}$$

Matlab Code: Lagrange_SCARA.m

Scientist Bio

Joseph-Louis Lagrange

(25 January 1736 – 10 April 1813)

Was an Italian mathematician and astronomer, later naturalized French. He made significant contributions to the fields of analysis, number theory, and both classical and celestial mechanics.

In 1766, on the recommendation of Swiss Leonhard Euler and French d'Alembert, Lagrange succeeded Euler as the director of mathematics at the Prussian Academy of Sciences in Berlin, Prussia, where he stayed for over twenty years, producing volumes of work and winning several prizes of the French Academy of Sciences. Lagrange's treatise on analytical mechanics written in Berlin and first published in 1788, offered the most comprehensive treatment of classical mechanics since Newton and formed a basis for the development of mathematical physics in the nineteenth century. In 1787, at age 51, he moved from Berlin to Paris and became a member of the French Academy of Sciences. He remained in France until the end of his life. He was instrumental in the decimalisation in Revolutionary France, became the first professor of analysis at the École Polytechnique upon its opening in 1794, was a founding member of the Bureau des Longitudes, and became Senator in 1799.

Lagrange was one of the creators of the calculus of variations, deriving the Euler–Lagrange equations for extrema of functionals. He extended the method to include possible constraints, arriving at the method of Lagrange multipliers. Lagrange invented the method of solving differential equations known as variation of parameters, applied differential calculus to the theory of probabilities and worked on solutions for algebraic equations. He proved that every natural number is a sum of four squares. His treatise *Theorie des fonctions analytiques* laid some of the foundations of group theory, anticipating Galois. In calculus, Lagrange developed a novel approach to interpolation and Taylor series. He studied the three-body problem for the Earth, Sun and Moon (1764) and the movement of Jupiter's satellites (1766), and in 1772 found the special-case solutions to this problem that yield what are now known as Lagrangian points. Lagrange is best known for transforming Newtonian mechanics into a branch of analysis, Lagrangian mechanics, and presented the mechanical "principles" as simple results of the variational calculus.

Contents

In this chapter we review the dynamics analysis for serial robots. First the definition to angular and linear accelerations are given, then mass properties, linear and angular momentums, and kinetic energy of a rigid body in space is defined. Lagrange formulation is given in general form, and dynamics mass matrix, gravity and Coriolis and centrifugal vectors are defined and derived for several case studies. Dynamic formulation properties is given next, then actuator dynamics is elaborated for electrically driven robots with gearbox. Finally, linear regression method is used for dynamics calibration, and model verification methods are elaborated by introducing consistency measure.

Dynamic Formulation Properties

- Dynamics Formulation Representation

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\mathbf{q}} + \mathbf{v}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \mathbf{g}(\boldsymbol{q}) = \boldsymbol{Q}$$

✓ Mass Matrix Properties

▪ Since the mass matrix is defined from kinetic energy: $K = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}$

It is always symmetric and positive definite (∀ Configurations $\boldsymbol{q}$)

It is always invertible

It has upper and lower bound

$$\underline{\lambda}\, \boldsymbol{I}_{n\times n} \leq \boldsymbol{M}(\boldsymbol{q}) \leq \bar{\lambda}\, \boldsymbol{I}_{n\times n}$$

Furthermore,

$$\frac{1}{\bar{\lambda}}\, \boldsymbol{I}_{n\times n} \leq \boldsymbol{M}^{-1}(\boldsymbol{q}) \leq \frac{1}{\underline{\lambda}} \boldsymbol{I}_{n\times n}$$

The bounds may be represented by matrix norm

$$\underline{M} \leq \|\boldsymbol{M}(\boldsymbol{q})\| \leq \bar{M}$$

In which $\underline{M}$ and $\bar{M}$ are positive constants.

Dynamic Formulation Properties

- Linearity in Parameters

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \mathbf{v}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \mathbf{g}(\boldsymbol{q}) = \boldsymbol{Q}$$

✓ This formulation is nonlinear and multivariable w.r.t $\boldsymbol{q}$

- But It could be written in linear regression form w.r.t kinematic and dynamic parameters

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \mathbf{v}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\mathcal{Y}}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\boldsymbol{\Phi}$$

In which, $\boldsymbol{\mathcal{Y}}$ denote the linear regressor form

While $\boldsymbol{\Phi}$ denote the kinematic and dynamic parameter vector

✓ This regression form is not unique

- But could be found for any serial robot
- The minimum number of parameters may found by inspection
- Or calibration process and singular values decomposition

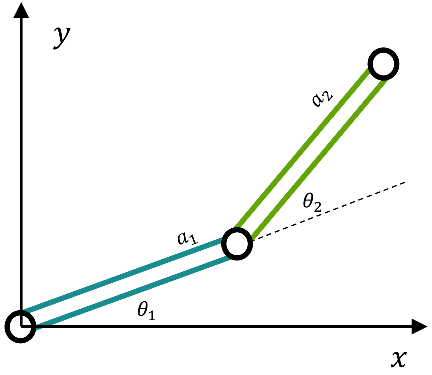
Dynamic Formulation Properties

- Linearity in Parameters
  - ✓ Example: Planar Elbow Manipulator
    - Dynamic formulation:

$$\tau_1 = \left[\frac{1}{3}m_1a_1^2 + m_2\left(a_1^2 + a_1a_2c_2 + \frac{1}{3}a_2^2\right)\right]\ddot{\theta}_1 + \left[m_2\left(\frac{1}{2}a_1a_2c_2 + \frac{1}{3}a_2^2\right)\right]\ddot{\theta}_2$$

$$- m_2a_1a_2s_2\left(\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}\dot{\theta}_2^2\right) + \left(\frac{1}{2}m_1 + m_2\right)ga_1c_1 + \frac{1}{2}m_2ga_2c_{12} + b_1\dot{\theta}_1$$

$$\tau_2 = \left[m_2\left(\frac{1}{2}a_1a_2c_2 + \frac{1}{3}a_2^2\right)\right]\ddot{\theta}_1 + \frac{1}{3}m_2a_2^2\ddot{\theta}_2 + \frac{1}{2}m_2a_1a_2s_2\dot{\theta}_1^2 + \frac{1}{2}m_2ga_2c_{12} + b_2\dot{\theta}_2$$



- Choose the parameters as:

$$\phi_1 = \tfrac{1}{3}m_1a_1^2 + m_2\left(a_1^2 + \tfrac{1}{3}a_2^2\right)$$
$$\phi_2 = m_2a_1a_2, \quad \phi_3 = \tfrac{1}{3}m_2a_2^2$$
$$\phi_4 = \left(\tfrac{1}{2}m_1 + m_2\right), \quad \phi_5 = \tfrac{1}{2}m_2a_2$$
$$\phi_6 = b_1, \ \phi_7 = b_2$$

$$\Rightarrow$$

$$M_{11} = \phi_1 + \phi_2c_2, \quad M_{22} = \phi_3$$
$$M_{12} = M_{21} = \tfrac{1}{2}\phi_2c_2 + \phi_3$$
$$g_1 = \phi_4gc_1 + \phi_5gc_{12}, \quad g_2 = \phi_5gc_{12}$$
$$v_1 = -\phi_2s_2\left(\dot{\theta}_1\dot{\theta}_2 + \tfrac{1}{2}\dot{\theta}_2^2\right), \ v_2 = \tfrac{1}{2}\phi_2s_2\dot{\theta}_1^2$$

Dynamic Formulation Properties

- Linearity in Parameters
  - ✓ Example: Planar Elbow Manipulator
    - Then

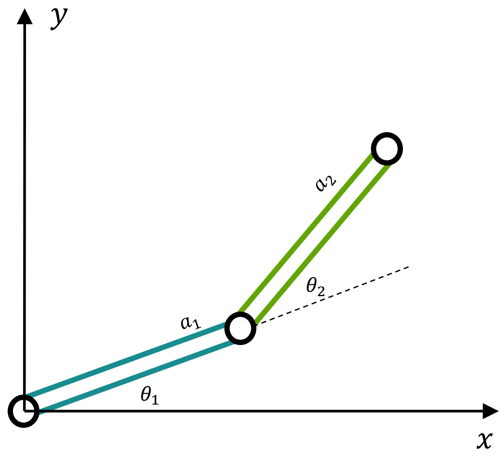
$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \mathbf{v}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \mathbf{g}(\boldsymbol{q}) = \boldsymbol{\mathcal{Y}}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\boldsymbol{\Phi}$$

    - In which

$$\boldsymbol{\mathcal{Y}} = \begin{bmatrix} \ddot{\theta}_1 & c_2\left(\ddot{\theta}_1 + \frac{1}{2}\ddot{\theta}_2\right) - s_2\left(\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}\dot{\theta}_2^2\right) & \ddot{\theta}_2 & gc_1 & gc_{12} & \dot{\theta}_1 & 0 \\ 0 & \frac{1}{2}\left(c_2\ddot{\theta}_1 + s_2\,\dot{\theta}_1^2\right) & \ddot{\theta}_1 + \ddot{\theta}_2 & 0 & gc_{12} & 0 & \dot{\theta}_2 \end{bmatrix},$$

and

$$\boldsymbol{\Phi} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \\ \phi_7 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}m_1a_1^2 + m_2\left(a_1^2 + \frac{1}{3}a_2^2\right) \\ m_2a_1a_2 \\ \frac{1}{3}m_2a_2^2 \\ \left(\frac{1}{2}m_1 + m_2\right) \\ \frac{1}{2}m_2a_2 \\ b_1 \\ b_2 \end{bmatrix}.$$



Dynamic Formulation Properties

- Christoffel Matrix
  - ✓ The Coriolis and centrifugal vector may be written in Matrix form:

$$\mathbf{v}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$$

- To derive this Matrix, recall Kronecker product:

For two matrices $\boldsymbol{A}_{n\times m}$, and $\boldsymbol{B}_{p\times r}$ $\rightarrow$ $\boldsymbol{A}\otimes\boldsymbol{B} = \left[a_{ij}\boldsymbol{B}\right]_{np\times mr}$

The matrix has $p\times r$ blocks, each determined by term-by-term multiplication of $a_{ij}\boldsymbol{B}$

Note that $\boldsymbol{I}_{n\times n}\otimes\boldsymbol{x} \neq \boldsymbol{x}\otimes\boldsymbol{I}_{n\times n}$ but $(\boldsymbol{I}_{n\times n}\otimes\boldsymbol{x})\boldsymbol{x} = (\boldsymbol{x}\otimes\boldsymbol{I}_{n\times n})\boldsymbol{x}$

For any arbitrary $\boldsymbol{x}\in\mathbb{R}^n$

Finally,

$$\frac{\partial A}{\partial q} = \begin{bmatrix} \frac{\partial A}{\partial q_1} \\ \vdots \\ \frac{\partial A}{\partial q_n} \end{bmatrix} \rightarrow \frac{\partial}{\partial q}\left(\boldsymbol{A}(\boldsymbol{q})\boldsymbol{B}(\boldsymbol{q})\right) = (\boldsymbol{I}_{n\times n}\otimes\boldsymbol{A})\frac{\partial B}{\partial q} + \frac{\partial A}{\partial q}\boldsymbol{B}.$$

Dynamic Formulation Properties

- Christoffel Matrix
  - ✓ The Coriolis and centrifugal vector is defined by

$$\mathbf{v}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\frac{\partial}{\partial \boldsymbol{q}}\left(\dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}\right)$$

  - ▪ Use Kronecker product, and note that $\frac{\partial \dot{\boldsymbol{q}}^T}{\partial \boldsymbol{q}} = 0$, then

$$\mathbf{v}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \left[\dot{\boldsymbol{M}}(\boldsymbol{q}) - \frac{1}{2}\left(\boldsymbol{I}_{n\times n}\otimes\dot{\boldsymbol{q}}^T\right)\frac{\partial \boldsymbol{M}}{\partial \boldsymbol{q}}\right]\dot{\boldsymbol{q}}.$$

  - ▪ Hence,

$$\mathbf{v}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{C}_1(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}$$

In which,

$$\boldsymbol{C}_1(\boldsymbol{q},\dot{\boldsymbol{q}}) = \dot{\boldsymbol{M}}(\boldsymbol{q}) - \frac{1}{2}\boldsymbol{U}(\boldsymbol{q},\dot{\boldsymbol{q}})$$

where,

$$\boldsymbol{U}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \left(\boldsymbol{I}_{n\times n}\otimes\dot{\boldsymbol{q}}^T\right)\frac{\partial \boldsymbol{M}}{\partial \boldsymbol{q}} = \begin{bmatrix} \dot{q}_1 & 0 & \cdots & 0 \\ 0 & \dot{q}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \dot{q}_n \end{bmatrix}\begin{bmatrix} \frac{\partial \boldsymbol{M}}{\partial q_1} & \frac{\partial \boldsymbol{M}}{\partial q_2} & \cdots & \frac{\partial \boldsymbol{M}}{\partial q_n} \end{bmatrix}.$$

Dynamic Formulation Properties

- Christoffel Matrix
  - This representation is not unique! Find another one.

$$\boldsymbol{C}_2(\boldsymbol{q},\dot{\boldsymbol{q}}) = \boldsymbol{U}^T(\boldsymbol{q},\dot{\boldsymbol{q}}) - \frac{1}{2}\boldsymbol{U}(\boldsymbol{q},\dot{\boldsymbol{q}})$$

  Notice $\dot{\boldsymbol{M}} \neq \boldsymbol{U}^T$ but $\dot{\boldsymbol{M}}\dot{\boldsymbol{q}} = \boldsymbol{U}^T\dot{\boldsymbol{q}}$.

  - The most celebrated Christofell Matrix is derived by:

$$\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \frac{1}{2}\left(\dot{\boldsymbol{M}}(\boldsymbol{q}) + \boldsymbol{U}^T(\boldsymbol{q},\dot{\boldsymbol{q}}) - \boldsymbol{U}(\boldsymbol{q},\dot{\boldsymbol{q}})\right)$$

- Skew-Symmetric Property
  - ✓ For all C_i's this relation holds

$$\dot{\boldsymbol{q}}^T\left(\dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\boldsymbol{C}_i(\boldsymbol{q},\dot{\boldsymbol{q}})\right)\dot{\boldsymbol{q}} = 0$$

  - ✓ But for $\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$ the own matrix $\dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$ is skew symmetric.

Dynamic Formulation Properties

- Christoffel Matrix
  - Proof:

1) $\dot{q}^T(\dot{M} - 2C_1)\dot{q} = \dot{q}^T(\dot{M} - 2\dot{M} + U)\dot{q} = (\dot{q}^T U - \dot{q}^T \dot{M})\dot{q} = 0$ since $\dot{M}\dot{q} = U^T\dot{q}$.

2) $\dot{q}^T(\dot{M} - 2C_2)\dot{q} = \dot{q}^T(\dot{M} - 2U^T + U)\dot{q} = \dot{q}^T(\dot{M}\dot{q} - (2U^T - U)\dot{q}) = \dot{q}^T(-U^T$

Dynamic Formulation Properties

- Passivity
  - ✓ Claim: the amount of energy dissipated by the robot has a negative lower bound $-\beta$

$$\int_0^T \dot{\boldsymbol{q}}^T(\xi)\boldsymbol{\tau}(\xi)d\xi \geq -\beta$$

- Note $\dot{\boldsymbol{q}}^T\boldsymbol{\tau}$ has the units of power, and the integral denotes the energy produced (or dissipated) by the system in the interval $[0, T]$
- How this is related to the rate of total energy $\dot{H}$:

In robotic manipulators:

Total Energy $H = K + P = \frac{1}{2}\dot{\boldsymbol{q}}^T\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}} + P(\boldsymbol{q})$

Dynamic Formulation Properties

- Passivity
  - Rate of total energy is:

$$\dot{H} = \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{\boldsymbol{q}}^T \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^{\boldsymbol{T}} \frac{\boldsymbol{\partial} P(\boldsymbol{q})}{\boldsymbol{\partial q}} = \dot{\boldsymbol{q}}^T\{\boldsymbol{\tau} - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \mathbf{g}(\boldsymbol{q})\} + \frac{1}{2}\dot{\boldsymbol{q}}^T \dot{\boldsymbol{M}}(\boldsymbol{q})\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^{\boldsymbol{T}} \frac{\boldsymbol{\partial} P(\boldsymbol{q})}{\boldsymbol{\partial q}}$$

$$= \dot{\boldsymbol{q}}^T \boldsymbol{\tau} + \frac{1}{2}\dot{\boldsymbol{q}}^T\{\dot{\boldsymbol{M}}(\boldsymbol{q}) - \boldsymbol{2C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\}\dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^T \left\{\frac{\boldsymbol{\partial} P(\boldsymbol{q})}{\boldsymbol{\partial q}} - \mathbf{g}(\boldsymbol{q})\right\} = \dot{\boldsymbol{q}}^T \boldsymbol{\tau}$$

The last two terms are both zero by skew-symmetric property and definition of gravity vector.

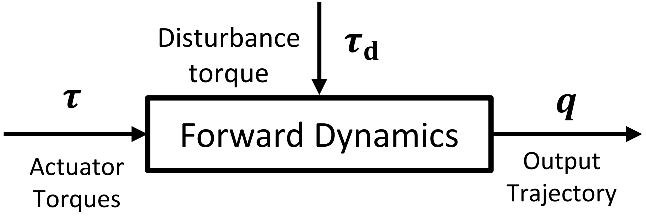
✓ Hence, because $\dot{H} = \dot{\boldsymbol{q}}^T \boldsymbol{\tau}$

$$\int_0^T \dot{H}(\xi)d\xi = \int_0^T \dot{\boldsymbol{q}}^T(\xi)\boldsymbol{\tau}(\xi)d\xi = H(T) - H(0) \geq -H(0)$$

  - Since the total energy of the system is always positive, for any robotic system the passivity property is held with $\beta = H(0)$.

Dynamic Simulation

- Forward Dynamics
  - ✓ In forward dynamics given the actuator forces applied to the robot, the resulting output trajectory of the robot is found.



  - General Dynamics:

$$\boldsymbol{\tau} + \boldsymbol{\tau}_d = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \mathbf{g}(\boldsymbol{q})$$

  - The Dynamics equations shall be integrated to find the trajectory
  - The mass matrix is positive definite, hence it is invertible in all configurations

$$\ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(\boldsymbol{q})\{\boldsymbol{\tau} + \boldsymbol{\tau}_d - \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} - \mathbf{g}(\boldsymbol{q})\}$$

  - Use numerical integration like Runge-Kutta method (ode45 in Matlab or Simulink)

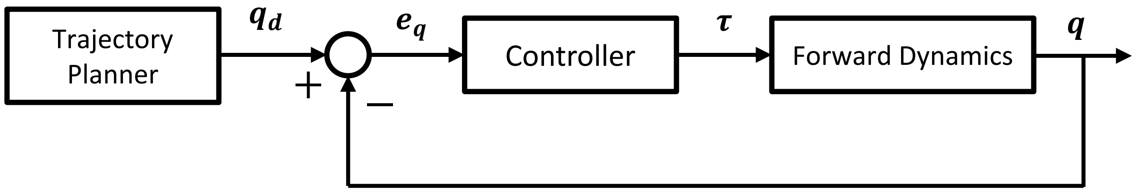
Given the torques $\boldsymbol{\tau}, \boldsymbol{\tau}_d$ and the initial conditions for the augmented states $\boldsymbol{x} = [\boldsymbol{x}_1, \boldsymbol{x}_2]^T = [\boldsymbol{q}, \dot{\boldsymbol{q}}]^T$, use numerical integration to solve for the trajectory.

$$\dot{\boldsymbol{x}}_1 = \dot{\boldsymbol{q}} = \boldsymbol{x}_2$$

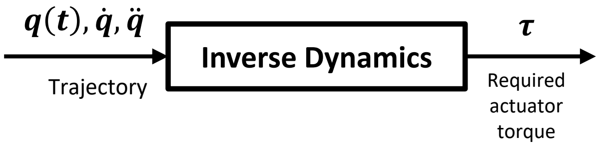
$$\dot{\boldsymbol{x}}_2 = \ddot{\boldsymbol{q}} = \boldsymbol{M}^{-1}(\boldsymbol{x}_1)\{\boldsymbol{\tau} + \boldsymbol{\tau}_d - \boldsymbol{C}(\boldsymbol{x}_1, \boldsymbol{x}_2)\boldsymbol{x}_2 - \mathbf{g}(\boldsymbol{x}_1)\}$$

Dynamic Simulation

- Forward Dynamics in Feedback loop
  - ✓ By using, trajectory planner, and controller design, a desired trajectory is traversed. (Chapter 6)



- Inverse Dynamics
  - ✓ In inverse dynamics given the trajectory of the robot $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t), \ddot{\boldsymbol{q}}(t))$, the required actuator forced to traverse this trajectory $\boldsymbol{\tau}$ is found.



- Use general dynamics:

$$\boldsymbol{\tau} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \mathbf{g}(\boldsymbol{q})$$

No integration is needed, direct calculation is possible.

Contents

In this chapter we review the dynamics analysis for serial robots. First the definition to angular and linear accelerations are given, then mass properties, linear and angular momentums, and kinetic energy of a rigid body in space is defined. Lagrange formulation is given in general form, and dynamics mass matrix, gravity and Coriolis and centrifugal vectors are defined and derived for several case studies. Dynamic formulation properties is given next, then actuator dynamics is elaborated for electrically driven robots with gearbox. Finally, linear regression method is used for dynamics calibration, and model verification methods are elaborated by introducing consistency measure.

Actuator Dynamics

- Robot Electrical Actuators

Permanent Magnet
DC Motors

Brushless DC Motors

AC Induction Motors

Actuator Dynamics

- DC Motors
  - ✓ Principle of operation
    - A current carrying conductor in a magnetic field experiences a Force

$$F = i \times \phi$$

i: the current in the conductor, ϕ: the magnetic field flux

    - DC motor consists of

A fixed Stator (Permanent magnet)

A rotating rotor (Armature)

A commutator: to switch the direction of current

    - The torque generated in the motor

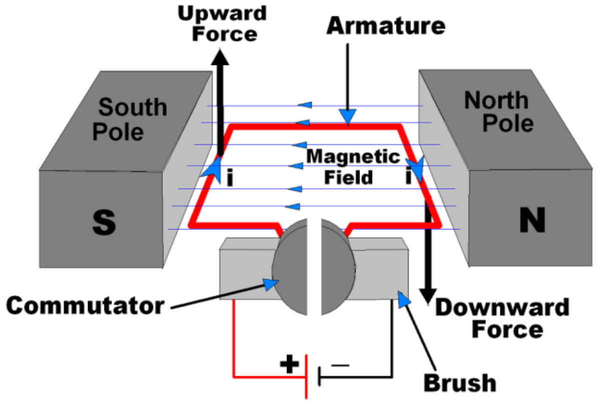
$$\tau_m = K_1 \phi i_a$$

i_a: the armature current, ϕ: the magnetic field flux

    - Lenz's Law: back-emf voltage

$$V_b = K_2 \phi \omega_m$$

ω_m: the angular velocity of the rotor, ϕ: the magnetic field flux



DC Motor Conceptual Diagram

Actuator Dynamics

- DC Motors
  - ✓ Principle of operation
    - For a permanent magnet DC motor ϕ is constant

      If SI unit is used:

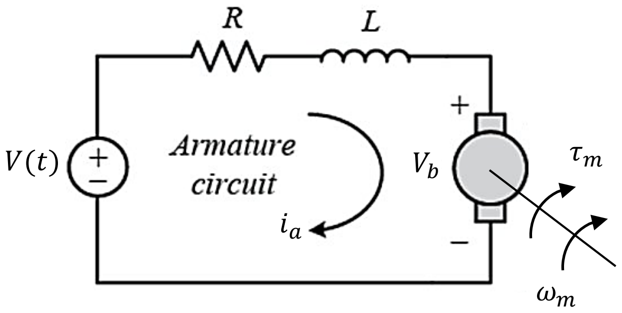
      $$\tau_m = K_m i_a \quad \text{and} \quad V_b = K_m \omega_m$$

      K_m: the DC motor torque or velocity constant
    - Electrical model of the PMDC motor

      $$L\frac{di_a}{dt} + Ri_a + V_b = V(t) \text{ or } (Ls+R)i_a = V - V_b$$

      $$i_a = \frac{V - V_b}{(Ls+R)}$$
    - Electro-mechanical model of the PMDC motor

      $$\tau_m = K_m i_a = \tau_m = K_m \frac{V - K_m\omega_m}{(Ls+R)}$$



Actuator Dynamics

- DC Motors
  - ✓ Principle of operation
    - Usually $\frac{L}{R} \ll 1 \; (\cong 10^{-2} - 10^{-3})$
    - Then up to $100 - 1000\ Hz$ bandwidth neglect induction

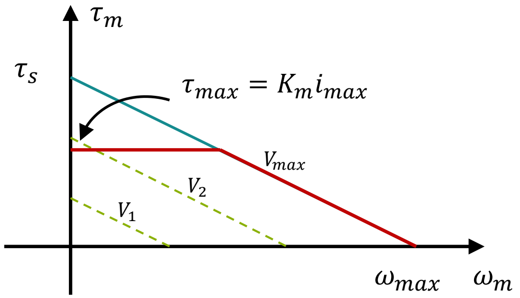
$$\tau_m = \frac{K_m}{R}(V - K_m \omega_m)$$

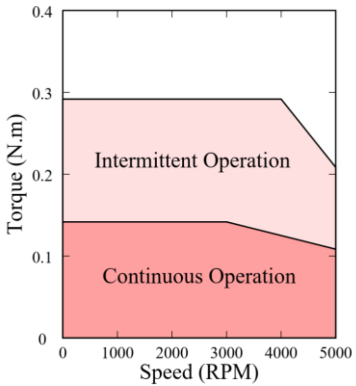
    - The Motor characteristic torque-velocity curve is linear
    - To avoid overheating i_a is clamped to i_{max} by current limiters
    - The stall torque τ_s is the maximum output torque of the motor, when the velocity of the motor is stalled.

$$V_s = \frac{R\tau_s}{K_m}$$

    - The maximum speed of the motor ω_{max} is obtained when no load torque is applied to the motor
    - Back-emf acts like a disturbance on the output torque, the more the velocity the less the output torque.

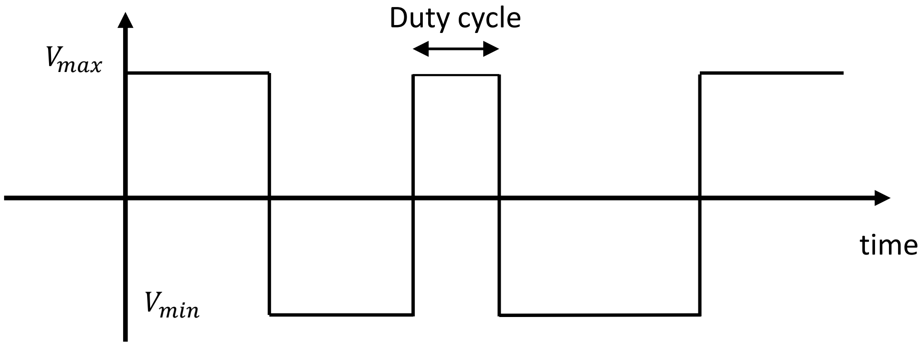
How to compensate?!!





Actuator Dynamics

- Servo Amplifier
  - ✓ Principle of operation
    - Voltage is generated in PWM (pulse-width modulation)



- If the duty cycle is $= 50\%$ $\rightarrow$ $v_{out} = 0$
- If the duty cycle is $> 50\%$ $\rightarrow$ $v_{out} > 0$
- If the duty cycle is $< 50\%$ $\rightarrow$ $v_{out} < 0$

The PWM frequency is usually $> 25Hz$

Actuator Dynamics

- Servo Amplifier
  - ✓ Principle of operation
    - Current (Torque) mode

      An internal current feedback with a tuned PI controller

      Ideally considered as a current (torque) source

      Back-emf effect limits the performance up to $100\ Hz$ bandwidth

      Command signal i_d is tracked within the bandwidth
    - Velocity (Voltage) Mode

      An external tacho (or velocity) feedback with a tuned lead controller

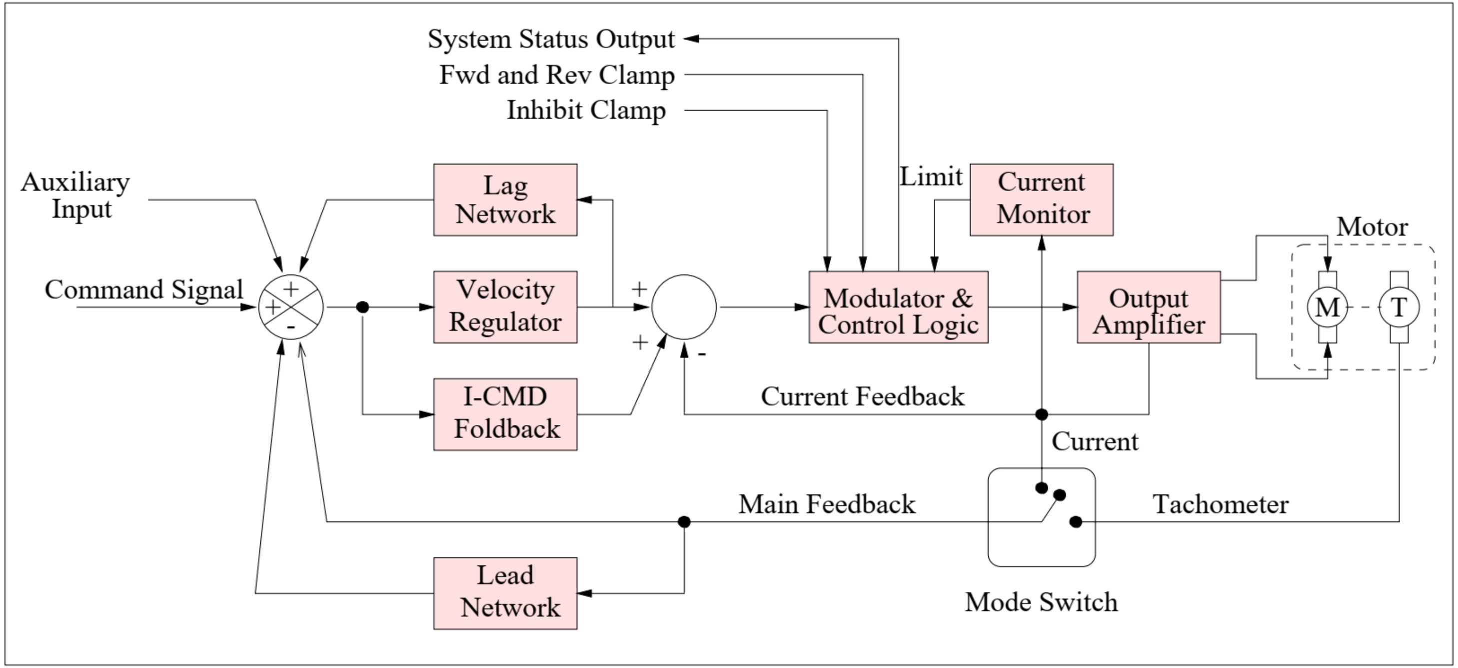
      Command signal ω_d is tracked within the bandwidth when switched to this mode
    - Current limiter with current foldback

      Usually $\pm i_{max}$ at continuous operation while permitting higher current intermittently

      FWD and REV current clamp

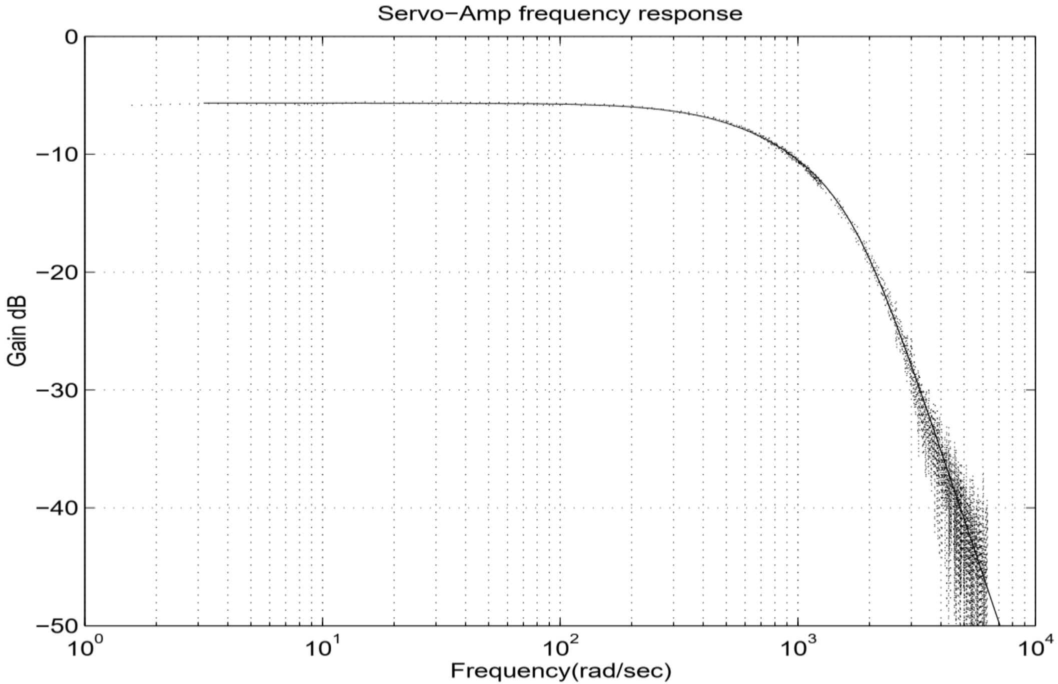
Actuator Dynamics

- Servo Amplifier (Function Diagram)



Actuator Dynamics

- Servo Amplifier (Current Mode Frequency Response)

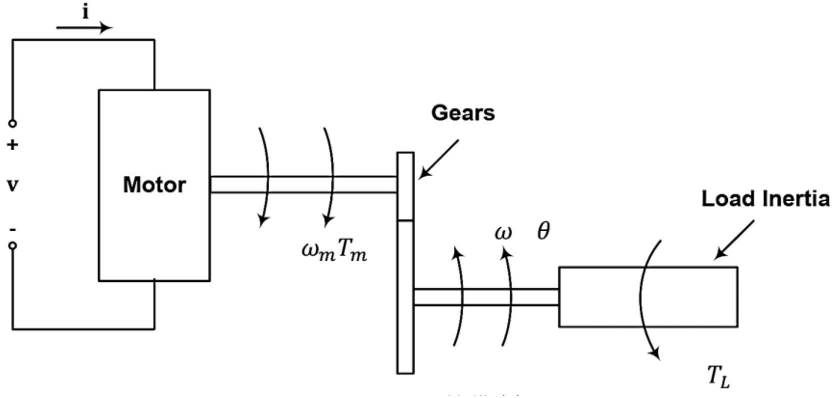


Actuator Dynamics

- Gearbox
  - ✓ Principle of operation
    - Electric motors provide low torque but have high velocity
    - Use Gear box to increase the torque while reducing the speed
    - Ideal Gearbox with ratio η

$$\theta_m = \eta\,\theta \quad \text{and} \quad \omega_m = \eta\,\omega \quad \text{while} \quad \tau_m = \frac{1}{\eta}\tau$$

By use principle of virtual work: $\tau_m \cdot \theta_m = \tau \cdot \theta$



Actuator Dynamics

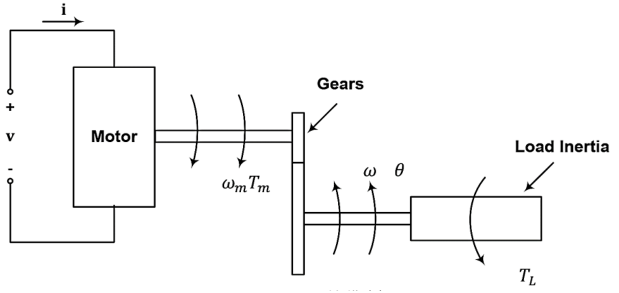
✓ Motor – Gearbox – Load Dynamics

- Notation: subscript m for motor side, no subscript for load side
- Suppose inertia I_m, I and viscous friction b_m, b

Motor side dynamics: Load side dynamics:

$$\tau_m = I_m\ddot{\theta}_m + b_m\dot{\theta}_m + \frac{1}{\eta}\tau,$$

$$\tau = I\ddot{\theta} + b\dot{\theta}$$



Overall dynamics:

$$\tau_m = I_m\ddot{\theta}_m + b_m\dot{\theta}_m + \frac{1}{\eta}(I\ddot{\theta} + b\dot{\theta})$$

Substitute: $\dot{\theta}_m = \eta\,\dot{\theta}$ and $\ddot{\theta}_m = \eta\,\ddot{\theta}$ and consider $\tau_m = \frac{1}{\eta}\tau$

@ motor side: $\tau_m = \left(I_m + \frac{1}{\eta^2}I\right)\ddot{\theta}_m + \left(b_m + \frac{1}{\eta^2}b\right)\dot{\theta}_m$

@ Load side: $\tau = (\eta^2 I_m + I)\,\ddot{\theta} + (\eta^2 b_m + b)\,\dot{\theta}$

Actuator Dynamics

✓ Motor – Gearbox – Multiple (n) DOFs Robot

The effect of motor inertia and damping is highlighted by a factor of η^2

Shorthand Dynamics:

$$\tau = I_e\,\ddot{\theta} + b_e\,\dot{\theta}$$

In which the effective inertia and viscous friction are:

$$I_e = (\eta^2 I_m + I) \quad \text{and} \quad b_e = (\eta^2 b_m + b)$$

- Consider the general model of the robot

$$\boldsymbol{\tau} - \boldsymbol{b}\dot{\boldsymbol{q}} = \boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \mathbf{g}(\boldsymbol{q})$$

- Add motor – gearbox dynamics

$$\boldsymbol{\tau} - \boldsymbol{b}_e\dot{\boldsymbol{q}} = \boldsymbol{M}_e\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \mathbf{g}(\boldsymbol{q})$$

In which:

$$\boldsymbol{M}_e(\boldsymbol{q}) = \left\{\boldsymbol{M}(\boldsymbol{q}) + \begin{bmatrix} \eta_1^2 I_{m_1} & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & \dots & \eta_n^2 I_{m_n} \end{bmatrix}\right\} \quad \text{and} \quad \boldsymbol{b}_e = \begin{bmatrix} b_1 + \eta^2 b_{m_1} \\ \vdots \\ b_n + \eta^2 b_{m_n} \end{bmatrix}$$

Actuator Dynamics

✓ Motor – Gearbox – Multiple (n) DOFs Robot

- If $\eta_i \gg 1 \Rightarrow \eta_i^2$ dominates and dynamic equations will be decoupled

$$\tau_i - b_{e_i}\dot{q}_i = \{M_{ii}(q) + \eta_i^2 I_{m_i}\}\ddot{q}_i + v_i(q,\dot{q}) + g_i(q)$$

- If $\eta_i \ggg 1 \Rightarrow \eta_i^2$ dominates and the dynamics becomes decoupled and linear

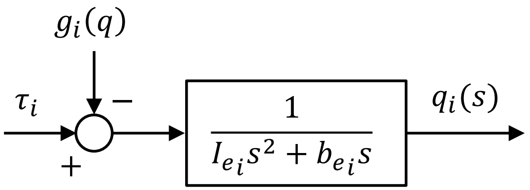
$$I_{e_i} = M_{ii} + \eta_i^2 I_{m_i} \approx \eta_i^2 I_{m_i}$$

$$b_{e_i} = b_i + \eta_i^2 b_{m_i} \approx \eta_i^2 b_{m_i}$$

Then $V_i(q,\dot{q}) + G_i(q)$ may become negligible compared to $I_{e_i}\ddot{q}_i + b_{e_i}\dot{q}_i$

- We usually keep the gravity term $g_i(q)$ but may neglect $v_i(q,\dot{q})$ then

$$\tau_i = I_{e_i}\ddot{q}_i + b_{e_i}\dot{q}_i + g_i(q)$$



Contents

In this chapter we review the dynamics analysis for serial robots. First the definition to angular and linear accelerations are given, then mass properties, linear and angular momentums, and kinetic energy of a rigid body in space is defined. Lagrange formulation is given in general form, and dynamics mass matrix, gravity and Coriolis and centrifugal vectors are defined and derived for several case studies. Dynamic formulation properties is given next, then actuator dynamics is elaborated for electrically driven robots with gearbox. Finally, linear regression method is used for dynamics calibration, and model verification methods are elaborated by introducing consistency measure.

Dynamics Calibration

- Linear Regression
  - ✓ Given the structure of the linear model for each joint

$$\tau_i = I_{e_i}\ddot{q}_i + b_{e_i}\dot{q}_i + g_i(q)$$

    - The dynamics parameters I_{e_i}, b_{e_i} and G_i may be identified by linear regression
    - Case (1)

Assume a set of experiments is done, and input torques τ, and output motions $q, \dot{q}, \ddot{q}$ for each joint are logged for a large number of samples

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_M \end{bmatrix}, \boldsymbol{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_M \end{bmatrix}, \ \dot{\boldsymbol{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_M \end{bmatrix}, \ \ddot{\boldsymbol{q}} = \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_M \end{bmatrix} \quad \text{for } i = 1,2,\ldots,M \text{ samples}$$

Assume the gravity term is set as a constant parameter $g_i(q) = g_i$.

The dynamic formulation for each link may be reformulated as a linear regression:

$$\boldsymbol{\tau} = I_{e_i}\ddot{\boldsymbol{q}} + b_{e_i}\dot{\boldsymbol{q}} + g_i = \boldsymbol{\mathcal{Y}}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\boldsymbol{\Phi} \quad \text{In which} \quad \boldsymbol{\mathcal{Y}}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = \begin{bmatrix} \ddot{q}_1 & \dot{q}_1 & 1 \\ \ddot{q}_2 & \dot{q}_2 & 1 \\ \vdots & \vdots & \vdots \\ \ddot{q}_M & \dot{q}_M & 1 \end{bmatrix}, \boldsymbol{\Phi} = \begin{bmatrix} I_{e_i} \\ b_{e_i} \\ g_i \end{bmatrix}, \boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_M \end{bmatrix}$$

Dynamics Calibration

- Linear Regression
  - ✓ The linear Regression $\boldsymbol{\tau} = \boldsymbol{\mathcal{Y}}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})\boldsymbol{\Phi}$ is an overdetermined set of equations
    - Find least-squares solution for the parameters by left-pseudo-inverse

$$\hat{\boldsymbol{\Phi}} = \boldsymbol{\mathcal{Y}}^{\dagger} \cdot \boldsymbol{\tau} \quad \text{in which} \quad \boldsymbol{\mathcal{Y}}^{\dagger} = (\boldsymbol{\mathcal{Y}}^{T}\boldsymbol{\mathcal{Y}})^{-1}\boldsymbol{\mathcal{Y}}^{\boldsymbol{T}}$$

    - Direct evaluation of $\boldsymbol{\mathcal{Y}}^{\dagger}$ might be intractable, use house holder reflection

Factorize $\boldsymbol{\mathcal{Y}}$ into $\boldsymbol{\mathcal{Y}}\boldsymbol{P} = \boldsymbol{Q}\boldsymbol{R}$

In which $\boldsymbol{P}_{n\times n}$ is a permutation matrix, $\boldsymbol{Q}_{M\times n}$ is an orthogonal matrix, and $\boldsymbol{R}_{n\times n}$ is upper triangular. Then

$$\hat{\boldsymbol{\Phi}} = \boldsymbol{P}\boldsymbol{R}^{-1}(\boldsymbol{Q}^{T}\boldsymbol{b})$$

And $\boldsymbol{R}^{-1}$ is calculated by back substitution.

    - Note: Matlab `pinv(Y)` calculates the right-pseudo inverse by householder reflection.

Dynamics Calibration

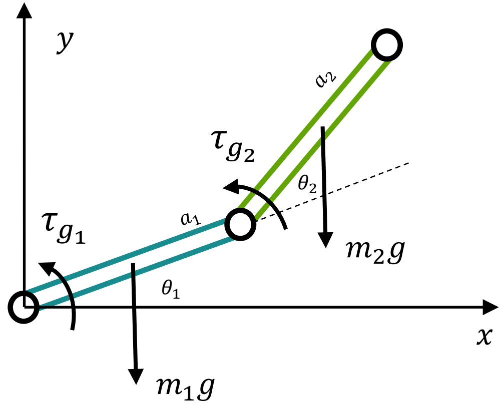
- Linear Regression
  - ✓ Given the structure of the linear model for each joint

$$\tau_i = I_{e_i}\ddot{q}_i + b_{e_i}\dot{q}_i + g_i(q)$$

  - Case (2):
  - Complete input-output logging and general gravity term $g_i(q)$.

In many examples $g_i(q) = g_i\cos(q)$.

$$\tau_{g_1} = \frac{1}{2}a_1 m_1 g\cos(\theta_1),\tau_{g_2} = \frac{1}{2}a_2 m_2 g\cos(\theta_1+\theta_2)$$



Then

$$\boldsymbol{\tau} = I_{e_i}\ddot{\boldsymbol{q}} + b_{e_i}\dot{\boldsymbol{q}} + g_i\cos(q) = \boldsymbol{\mathcal{Y}}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})\boldsymbol{\Phi}$$

In which

$$\boldsymbol{\mathcal{Y}}(\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}) = \begin{bmatrix} \ddot{q}_1 & \dot{q}_1 & \cos q_1 \\ \ddot{q}_2 & \dot{q}_2 & \cos q_2 \\ \vdots & \vdots & \vdots \\ \ddot{q}_M & \dot{q}_M & \cos q_M \end{bmatrix}, \boldsymbol{\Phi} = \begin{bmatrix} I_{e_i} \\ b_{e_i} \\ g_i \end{bmatrix}, \boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_M \end{bmatrix}$$

And therefore,

$$\widehat{\boldsymbol{\Phi}} = \boldsymbol{\mathcal{Y}}^{\dagger}\cdot\boldsymbol{\tau} \quad \text{or} \quad \widehat{\boldsymbol{\Phi}} = \texttt{pinv}(\boldsymbol{\mathcal{Y}}) * \boldsymbol{\tau}$$

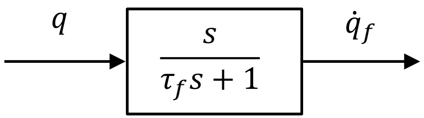
Dynamics Calibration

- Linear Regression
  - ✓ Given the structure of the linear model for each joint

$$\tau_i = I_{e_i}\ddot{q}_i + b_{e_i}\dot{q}_i + g_i(q)$$

    - Case (3): Only $\boldsymbol{\tau}$ and $\boldsymbol{q}$ are measured

Use filtered differentiation



In which

$\omega_f(Hz) = \frac{1}{\tau_f}$ is selected such that $10\ \omega_{BW} < \omega_f < 0.1\ \omega_{noise}$

Where ω_{BW} denotes the system bandwidth frequency while ω_{noise} denotes the major noise frequency content

Practically $0.001 < \tau_f < 0.05$ or equivalently $20 < \omega_f < 10^3$ would be a good choice.

Use Matlab command `filtfilt` to remove the delay in the filtered signal

Use the filter twice to find the angular acceleration $\ddot{q}_f$.

Check the filtered differentiation output and tune τ_f to reduce the output noise.

Dynamics Calibration

- Model Verification
  - ✓ Note that the simplified linear model is good
    - For linear controller design
    - Shall be valid for different operating regimes
    - If $\eta > 100$ the linear term dominates the nonlinear vector $\mathbf{v}(q, \dot{q})$
  - ✓ Verify the calibrated dynamic parameters
    - For different experiments by statistical analysis

	Exp 1	Exp 2	. . .	Exp 10	AVG	Consistency Measure
I_e	${}^{1}I_e$	${}^{2}I_e$	. . .	${}^{10}I_e$	$I_{e_{avg}}$	STDEV / $I_{e_{avg}}$
b_e	${}^{1}b_e$	${}^{2}b_e$	. . .	${}^{10}b_e$	$b_{e_{avg}}$	STDEV / $b_{e_{avg}}$
g	${}^{1}g$	${}^{2}g$	. . .	${}^{10}g$	g_{avg}	STDEV / G_{avg}

Dynamics Calibration

- Model Verification
  - ✓ Verify the calibrated dynamic parameters
    - For different experiments by statistical analysis

      If C.M. $< 30\%$ The average parameter is suitable for controller design

      If C.M. $< 80\%$ The averaged parameters could be used for controller design using robust linear controllers

      If C.M. $> 80\%$ Then your model is incomplete and you need to add terms in your model

    - If for different experiments the obtained parameters are physically inconsistent

      For example you get negative moment of inertia or damping parameters

      This means your model is still not good enough for calibration

      You may use constrained optimization to add bound on the parameters

      Matlab `fmincon` command may be used in this case.

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About Hamid D. Taghirad

Hamid D. Taghirad has received his B.Sc. degree in mechanical engineering from Sharif University of Technology, Tehran, Iran, in 1989, his M.Sc. in mechanical engineering in 1993, and his Ph.D. in electrical engineering in 1997, both from McGill University, Montreal, Canada. He is currently the University Vice-Chancellor for Global strategies and International Affairs, Professor and the Director of the Advanced Robotics and Automated System (ARAS), Department of Systems and Control, Faculty of Electrical Engineering, K. N. Toosi University of Technology, Tehran, Iran. He is a senior member of IEEE, and Editorial board of International Journal of Robotics: Theory and Application, and International Journal of Advanced Robotic Systems. His research interest is *robust* and *nonlinear control* applied to *robotic systems*. His publications include five books, and more than 250 papers in international Journals and conference proceedings.

Robotics: Mechanics & Control

Chapter 5: Dynamic Analysis

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Thank You