Question 1 – first solution)

Given the rotation axis and angle $s = \theta\hat{s}$, the rotation matrix becomes:

$$ {}^{A}R_B = \begin{bmatrix} s_x^2(1-c\theta)+c\theta & s_xs_y(1-c\theta)-s_zs\theta & s_xs_z(1-c\theta)+s_ys\theta \\ s_ys_x(1-c\theta)+s_zs\theta & s_y^2(1-c\theta)+c\theta & s_ys_z(1-c\theta)-s_xs\theta \\ s_zs_x(1-c\theta)-s_ys\theta & s_zs_y(1-c\theta)+s_xs\theta & s_z^2(1-c\theta)+c\theta \end{bmatrix} $$

Defining unit quaternion as:

$$ \epsilon_1 = s_x\,sin\left(\frac{\theta}{2}\right), \epsilon_2 = s_y\,sin\left(\frac{\theta}{2}\right), \epsilon_3 = s_z\,sin\left(\frac{\theta}{2}\right), \epsilon_4 = cos\left(\frac{\theta}{2}\right) $$

We get $\epsilon_1^2+\epsilon_2^2+\epsilon_3^2+\epsilon_4^2 = 1$ and using trigonometric relations:

$$ sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1-cos\,\theta) $$
$$ cos^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1+cos\,\theta) $$
$$ sin\,\theta = 2\,sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right) $$

We derive:

$$ s_x = \frac{\epsilon_1}{sin\left(\frac{\theta}{2}\right)} \Rightarrow s_x^2 = \frac{\epsilon_1^2}{sin^2\left(\frac{\theta}{2}\right)} = \frac{2\epsilon_1^2}{(1-cos\,\theta)} $$
$$ s_y = \frac{\epsilon_2}{sin\left(\frac{\theta}{2}\right)} \Rightarrow s_y^2 = \frac{\epsilon_2^2}{sin^2\left(\frac{\theta}{2}\right)} = \frac{2\epsilon_2^2}{(1-cos\,\theta)} $$
$$ s_z = \frac{\epsilon_3}{sin\left(\frac{\theta}{2}\right)} \Rightarrow s_z^2 = \frac{\epsilon_3^2}{sin^2\left(\frac{\theta}{2}\right)} = \frac{2\epsilon_3^2}{(1-cos\,\theta)} $$

$$ cos\,\theta = 2\,cos^2\left(\frac{\theta}{2}\right) - 1 \Rightarrow cos\,\theta = 2\epsilon_4^2 - 1 $$

Using foregoing relations, screw axis rotation matrix ${}^{A}R_B$ elements can be expressed in terms of unit quaternion elements as:

$$ r_{11} = s_x^2(1-c\theta)+c\theta = 2\epsilon_1^2+2\epsilon_4^2-1 = 2-2\epsilon_2^2+2\epsilon_3^2-1 = 1-2\epsilon_2^2+2\epsilon_3^2 $$
$$ r_{12} = s_xs_y(1-c\theta)-s_zs\theta = \frac{\epsilon_1\epsilon_2}{sin^2\left(\frac{\theta}{2}\right)}(1-c\theta) - \frac{\epsilon_3}{sin\left(\frac{\theta}{2}\right)}\cdot 2\,sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right) = 2\epsilon_1\epsilon_2 - 2\epsilon_3\epsilon_4 $$

$$r_{13} = s_x s_z(1 - c\theta) + s_y s\theta = \frac{\epsilon_1\epsilon_3}{sin^2\left(\frac{\theta}{2}\right)}(1 - c\theta) + \frac{\epsilon_2}{sin\left(\frac{\theta}{2}\right)} \cdot 2\,sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right) = 2\epsilon_1\epsilon_3 + 2\epsilon_2\epsilon_4$$
$$r_{23} = s_y s_z(1 - c\theta) - s_x s\theta = \frac{\epsilon_2\epsilon_3}{sin^2\left(\frac{\theta}{2}\right)}(1 - c\theta) - \frac{\epsilon_1}{sin\left(\frac{\theta}{2}\right)} \cdot 2\,sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right) = 2\epsilon_2\epsilon_3 - 2\epsilon_1\epsilon_4$$

Likewise, we derive other elements:

$$r_{22} = 1 - 2\epsilon_1^2 - 2\epsilon_3^2$$
$$r_{33} = 1 - 2\epsilon_1^2 - 2\epsilon_2^2$$
$$r_{21} = 2\epsilon_1\epsilon_2 + 2\epsilon_3\epsilon_4$$
$$r_{31} = 2\epsilon_1\epsilon_3 - 2\epsilon_2\epsilon_4$$
$$r_{32} = 2\epsilon_2\epsilon_3 + 2\epsilon_1\epsilon_4$$

Therefore, ${}^{A}R_B$ in terms of unit quaternion elements becomes:

$$^{A}R_B = \begin{bmatrix} 1 - 2\epsilon_2^2 - 2\epsilon_3^2 & 2(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_4) & 2(\epsilon_1\epsilon_3 + \epsilon_2\epsilon_4) \\ 2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4) & 1 - 2\epsilon_1^2 - 2\epsilon_3^2 & 2(\epsilon_2\epsilon_3 - \epsilon_1\epsilon_4) \\ 2(\epsilon_1\epsilon_3 - \epsilon_2\epsilon_4) & 2(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4) & 1 - 2\epsilon_1^2 - 2\epsilon_2^2 \end{bmatrix}$$

Question 1 – second solution)

A quaternion can be expressed in the following form:

$$\epsilon = \epsilon_1 i + \epsilon_2 j + \epsilon_3 k + \epsilon_4$$

Regarding $p = (p_x i + p_y j + p_z k)$ is a vector expressed in quaternion number system and $p' = (p'_x i + p'_y j + p'_z k)$ is rotated vector of p around unit quaternion $\epsilon = (\epsilon_1 i + \epsilon_2 j + \epsilon_3 k + \epsilon_4)$.

The operator that maps p to p' in the quaternion number system is $p' = L_\epsilon(p) = \epsilon p \epsilon^*$.

$$p' = \epsilon p \epsilon^* = (\epsilon_1 i + \epsilon_2 j + \epsilon_3 k + \epsilon_4)(p_x i + p_y j + p_z k)(-\epsilon_1 i - \epsilon_2 j - \epsilon_3 k + \epsilon_4)$$
$$= \left[(1 - 2\epsilon_2^2 - 2\epsilon_3^2)p_x + (2\epsilon_1\epsilon_2 - 2\epsilon_3\epsilon_4)p_y + (2\epsilon_1\epsilon_3 + 2\epsilon_2\epsilon_4)p_z\right]i$$
$$+\left[(2\epsilon_1\epsilon_2 + 2\epsilon_3\epsilon_4)p_x + (1 - 2\epsilon_1^2 - 2\epsilon_3^2)p_y + (2\epsilon_2\epsilon_3 - 2\epsilon_1\epsilon_4)p_z\right]j$$
$$+\left[(2\epsilon_1\epsilon_3 - 2\epsilon_2\epsilon_4)p_x + (2\epsilon_2\epsilon_3 + 2\epsilon_1\epsilon_4)p_y + (1 - 2\epsilon_1^2 - 2\epsilon_2^2)p_z\right]k$$

By expressing preceding terms in matrix form:

$$\begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = \begin{bmatrix} 1 - 2\epsilon_2^2 - 2\epsilon_3^2 & 2(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_4) & 2(\epsilon_1\epsilon_3 + \epsilon_2\epsilon_4) \\ 2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4) & 1 - 2\epsilon_1^2 - 2\epsilon_3^2 & 2(\epsilon_2\epsilon_3 - \epsilon_1\epsilon_4) \\ 2(\epsilon_1\epsilon_3 - \epsilon_2\epsilon_4) & 2(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4) & 1 - 2\epsilon_1^2 - 2\epsilon_2^2 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

And in respect of $p' = {}^{A}R_B p$, rotation matrix in quaternion number system will be:

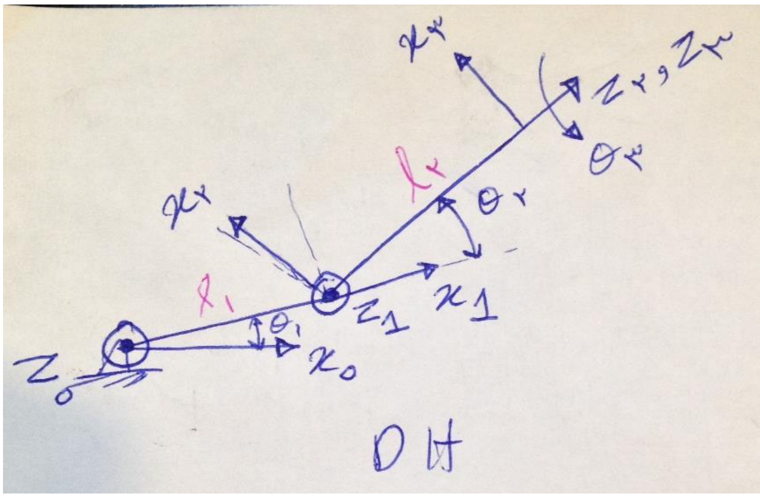
$$^{A}R_B = \begin{bmatrix} 1 - 2\epsilon_2^2 - 2\epsilon_3^2 & 2(\epsilon_1\epsilon_2 - \epsilon_3\epsilon_4) & 2(\epsilon_1\epsilon_3 + \epsilon_2\epsilon_4) \\ 2(\epsilon_1\epsilon_2 + \epsilon_3\epsilon_4) & 1 - 2\epsilon_1^2 - 2\epsilon_3^2 & 2(\epsilon_2\epsilon_3 - \epsilon_1\epsilon_4) \\ 2(\epsilon_1\epsilon_3 - \epsilon_2\epsilon_4) & 2(\epsilon_2\epsilon_3 + \epsilon_1\epsilon_4) & 1 - 2\epsilon_1^2 - 2\epsilon_2^2 \end{bmatrix}$$

Question 2)

1. Forward Kinematics:

Method1: DH parameters:

	a_i	α_i	d_i	θ_i
$i=1$	l_1	0	0	θ_1
$i=2$	0	0	0	$\theta_2+\pi/2$
$i=3$	0	$\pi/2$	l_2	θ_3



So:

$$^0T_E = \ ^0T_1\ ^1T_2\ ^2T_3$$

$$T_E = \begin{bmatrix} -\sin(\theta_1+\theta_2)\cos(\theta_3) & \sin(\theta_1+\theta_2)\sin(\theta_3) & \cos(\theta_1+\theta_2) & \cos(\theta_1+\theta_2)\,l_2 + l_1\cos(\theta_1) \\ \cos(\theta_1+\theta_2)\cos(\theta_3) & -\cos(\theta_1+\theta_2)\sin(\theta_3) & \sin(\theta_1+\theta_2) & \sin(\theta_1+\theta_2)\,l_2 + l_1\sin(\theta_1) \\ \sin(\theta_3) & \cos(\theta_3) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

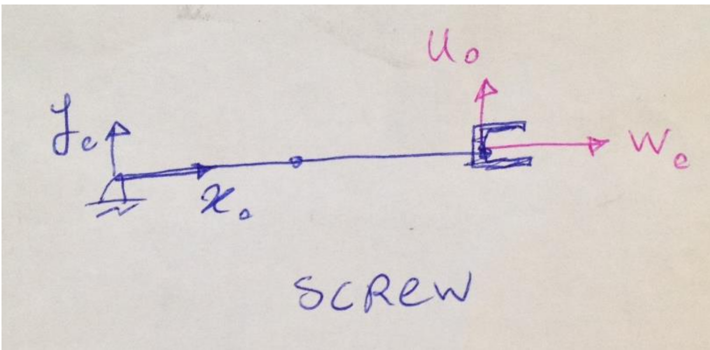
$$P := \begin{bmatrix} \cos(\theta_1+\theta_2)\,l_2 + l_1\cos(\theta_1) \\ \sin(\theta_1+\theta_2)\,l_2 + l_1\sin(\theta_1) \\ 0 \end{bmatrix}$$

$$R := \begin{bmatrix} -\sin(\theta_1+\theta_2)\cos(\theta_3) & \sin(\theta_1+\theta_2)\sin(\theta_3) & \cos(\theta_1+\theta_2) \\ \cos(\theta_1+\theta_2)\cos(\theta_3) & -\cos(\theta_1+\theta_2)\sin(\theta_3) & \sin(\theta_1+\theta_2) \\ \sin(\theta_3) & \cos(\theta_3) & 0 \end{bmatrix}$$

Method2: Screw parameters:

	s_i	s_{o_i}
$i=1$	$[0,0,1]^T$	$[0,0,0]^T$
$i=2$	$[0,0,1]^T$	$[a_1,0,0]^T$
$i=3$	$[1,0,0]^T$	$[a_1+a_2,0,0]^T$

$u_0 = [0,1,0]^T, v_0 = [0,0,1]^T, w_0 = [1,0,0]^T, P_0 = [a_1+a_2,0,0]^T$



So:

$$SFINAL := \begin{bmatrix} -\sin(\theta_1+\theta_2)\cos(\theta_3) & \sin(\theta_1+\theta_2)\sin(\theta_3) & \cos(\theta_1+\theta_2) & \cos(\theta_1+\theta_2)a_2 + a_1\cos(\theta_1) \\ \cos(\theta_1+\theta_2)\cos(\theta_3) & -\cos(\theta_1+\theta_2)\sin(\theta_3) & \sin(\theta_1+\theta_2) & \sin(\theta_1+\theta_2)a_2 + a_1\sin(\theta_1) \\ \sin(\theta_3) & \cos(\theta_3) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Inverse Kinematics:

For θ_1 and θ_2 , use 2-RR serial robot solution and $\theta_3 = atan2(R(3,1), R(3,2))$

3. Jacobian:

Method1: DH parameters:

$z_0 = [0,0,1]^T$

$z_1 = R_1[0,0,1]^T$

$z_2 = R_2[0,0,1]^T$

$J_w = [z_0, z_1, z_2]$

$${}^2p_3 = \begin{bmatrix} a_2cos(\theta_1+\theta_2) \\ a_2sin(\theta_1+\theta_2) \\ 0 \end{bmatrix}$$

$${}^1p_3 = \begin{bmatrix} a_2cos(\theta_1+\theta_2) \\ a_2sin(\theta_1+\theta_2) \\ 0 \end{bmatrix}$$

$${}^0p_3 = \begin{bmatrix} a_2cos(\theta_1+\theta_2) + a_1cos(\theta_1) \\ a_2sin(\theta_1+\theta_2) + a_1sin(\theta_1) \\ 0 \end{bmatrix}$$

$$J_v = [z_0 \times {}^0p_3 \quad z_1 \times {}^1p_3 \quad z_2 \times {}^2p_3]$$

Finally:

$$J = \begin{bmatrix} J_w \\ J_v \end{bmatrix}$$

$$J_{DH} := \begin{bmatrix} 0 & 0 & \cos(\theta_1+\theta_2) \\ 0 & 0 & \sin(\theta_1+\theta_2) \\ 1 & 1 & 0 \\ -\sin(\theta_1+\theta_2)\,a_2 - \sin(\theta_1)\,a_1 & -\sin(\theta_1+\theta_2)\,a_2 & 0 \\ \cos(\theta_1+\theta_2)\,a_2 + \cos(\theta_1)\,a_1 & \cos(\theta_1+\theta_2)\,a_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Method2: Screw parameters:

$i = 4, s_4 = [0,0,1]^T, s_{o4} = [0,0,0]^T$

$i = 3, s_3 = R_2[0,0,1]^T, s_{o3} = s_{o_4} - R_3[0,0,0]^T$

$i = 2, s_2 = R_1[0,0,1]^T, s_{o2} = s_{o_3} - R_2[0,0,0]^T$

$i = 1, s_1 = [0,0,1]^T, s_{o1} = s_{o_2} - R_1[0,0,0]^T$

$$J = \begin{bmatrix} s_1 & s_2 & s_3 \\ s_{o_1} \times s_1 & s_{o_2} \times s_2 & s_{o_3} \times s_3 \end{bmatrix}$$

$$J_{SR} := \begin{bmatrix} 0 & 0 & \cos(\theta_1+\theta_2) \\ 0 & 0 & \sin(\theta_1+\theta_2) \\ 1 & 1 & 0 \\ -\sin(\theta_1+\theta_2)\,a_2 - \sin(\theta_1)\,a_1 & -\sin(\theta_1+\theta_2)\,a_2 & 0 \\ \cos(\theta_1+\theta_2)\,a_2 + \cos(\theta_1)\,a_1 & \cos(\theta_1+\theta_2)\,a_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4. Velocities:

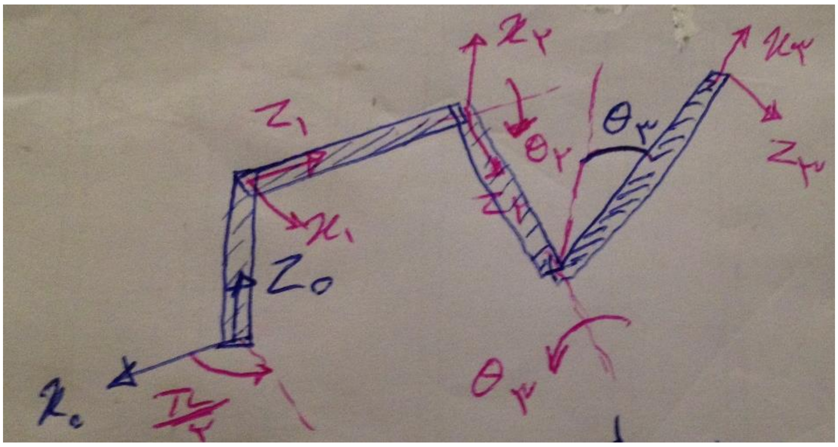
$\dot{q} = [\dot{q}_1, \dot{q}_2, \dot{q}_3]^T$

$Generelized\ Velocity = J\,\dot{q}$

$$Generelized\_Velocity := \begin{bmatrix} \cos(\theta_1 + \theta_2)\, \dot{q_3} \\ \sin(\theta_1 + \theta_2)\, \dot{q_3} \\ \dot{q_1} + \dot{q_2} \\ (-\sin(\theta_1 + \theta_2)\, a_2 - \sin(\theta_1)\, a_1)\, \dot{q_1} - \sin(\theta_1 + \theta_2)\, a_2\, \dot{q_2} \\ (\cos(\theta_1 + \theta_2)\, a_2 + \cos(\theta_1)\, a_1)\, \dot{q_1} + \cos(\theta_1 + \theta_2)\, a_2\, \dot{q_2} \\ 0 \end{bmatrix}$$

Question 3)

	a_i	α_i	d_i	θ_i
$i = 1$	l_1	$-\pi/2$	d_1	$\theta_1 + \pi/2$
$i = 2$	0	$-\pi/2$	d_2	$\theta_2 - \pi/2$
$i = 3$	a_3	0	d_3	$-\theta_3$



So:

$$P := \begin{bmatrix} -\sin(\theta_1)\sin(\theta_2)\, a_3 \cos(\theta_3) - \cos(\theta_1)\, a_3 \sin(\theta_3) - \sin(\theta_1)\cos(\theta_2)\, d_3 - \cos(\theta_1)\, d_2 \\ \cos(\theta_1)\sin(\theta_2)\, a_3\cos(\theta_3) - \sin(\theta_1)\, a_3 \sin(\theta_3) + \cos(\theta_1)\cos(\theta_2)\, d_3 - \sin(\theta_1)\, d_2 \\ \cos(\theta_2)\, a_3 \cos(\theta_3) - \sin(\theta_2)\, d_3 + d_1 \end{bmatrix} :$$

$$R := \begin{bmatrix} -\sin(\theta_1)\sin(\theta_2)\cos(\theta_3) - \cos(\theta_1)\sin(\theta_3) & -\sin(\theta_1)\sin(\theta_2)\sin(\theta_3) + \cos(\theta_1)\cos(\theta_3) & -\sin(\theta_1)\cos(\theta_2) \\ \cos(\theta_1)\sin(\theta_2)\cos(\theta_3) - \sin(\theta_1)\sin(\theta_3) & \cos(\theta_1)\sin(\theta_2)\sin(\theta_3) + \sin(\theta_1)\cos(\theta_3) & \cos(\theta_1)\cos(\theta_2) \\ \cos(\theta_2)\cos(\theta_3) & \cos(\theta_2)\sin(\theta_3) & -\sin(\theta_2) \end{bmatrix} :$$

Linear part of Jacobian matrix:

$$J_v = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} & \frac{\partial x}{\partial \theta_3} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} & \frac{\partial y}{\partial \theta_3} \\ \frac{\partial z}{\partial \theta_1} & \frac{\partial z}{\partial \theta_2} & \frac{\partial z}{\partial \theta_3} \end{bmatrix}$$

with(VectorCalculus);

Jv := Jacobian([X, Y, Z], [theta1, theta2, theta3]);

unwith(VectorCalculus);

$$J_V(1..3,1) := \begin{bmatrix} -\cos(\theta_1)\sin(\theta_2)\,a_3\cos(\theta_3) + \sin(\theta_1)\,a_3\sin(\theta_3) - \cos(\theta_1)\cos(\theta_2)\,d_3 + \sin(\theta_1)\,d_2 \\ -\sin(\theta_1)\sin(\theta_2)\,a_3\cos(\theta_3) - \cos(\theta_1)\,a_3\sin(\theta_3) - \sin(\theta_1)\cos(\theta_2)\,d_3 - \cos(\theta_1)\,d_2 \\ 0 \end{bmatrix}$$

$$J_V(1..3,2) := \begin{bmatrix} -\sin(\theta_1)\cos(\theta_2)\,a_3\cos(\theta_3) + \sin(\theta_1)\sin(\theta_2)\,d_3 \\ \cos(\theta_1)\cos(\theta_2)\,a_3\cos(\theta_3) - \cos(\theta_1)\sin(\theta_2)\,d_3 \\ -\sin(\theta_2)\,a_3\cos(\theta_3) - \cos(\theta_2)\,d_3 \end{bmatrix}:$$

$$J_V(1..3,3) := \begin{bmatrix} \sin(\theta_1)\sin(\theta_2)\,a_3\sin(\theta_3) - \cos(\theta_1)\,a_3\cos(\theta_3) \\ -\cos(\theta_1)\sin(\theta_2)\,a_3\sin(\theta_3) - \sin(\theta_1)\,a_3\cos(\theta_3) \\ -\cos(\theta_2)\,a_3\sin(\theta_3) \end{bmatrix}:$$

$\det(J_v) = 0 \rightarrow -cos(\theta_3)d_2(sin(\theta_2)a_3cos(\theta_3) + cos(\theta_2)d_3)a_3 \;=\; 0$

$cos(\theta_3) = 0 \rightarrow \theta_3 \neq \frac{k\pi}{2}$

$sin(\theta_2)a_3cos(\theta_3) + cos(\theta_2)d_3 = 0 \rightarrow \theta_2 \neq -\arctan\left(\frac{d_3}{a_3\cos(\theta_3)}\right)$